cse371/math371
LOGIC

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LECTURE 7
Chapter 7
Introduction to Intuitionistic and Modal Logics

PART 1: Intuitionistic Logic: Philosophical Motivation
Intuitionistic Logic: Philosophical Motivation

Intuitionistic logic has developed as a result of certain philosophical views on the foundation of mathematics, known as intuitionism. Intuitionism was originated by L. E. J. Brouwer in 1908.

The first Hilbert style formalization of the intuitionistic logic, formulated as a proof system, is due to A. Heyting (1930). We present a Hilbert style proof system $I$ that is equivalent to the Heyting’s original formalization.

We also discuss the relationship between intuitionistic and classical logic.
Intuitionistic Logic: Philosophical Motivation

There have been several successful attempts at creating semantics for the intuitionistic logic. The most recent called Kripke models were defined by Kripke in 1964.

The first intuitionistic semantics was defined in a form of pseudo-Boolean algebras by McKinsey and Tarski in years 1944 - 1946. Their algebraic approach to intuitionistic and classical semantics was followed by many authors and developed into a new field of Algebraic Logic.

The pseudo-Boolean algebras are called also Heyting algebras to memorize his first accepted formalization of the intuitionistic logic as a proof system.
Intuitionistic Logic: Philosophical Motivation

An uniform presentation of algebraic models for classical, intuitionistic and modal logics S4, S5 was first given in a now classic algebraic logic book:

"Mathematics of Metamathematics", Rasiowa, Sikorski (1964)

The main goal of this chapter is to give a presentation of the intuitionistic logic formulated as Hilbert and Gentzen proof systems.

We also discuss its algebraic semantics and the fundamental theorems that establish the relationship between classical and intuitionistic propositional logics.
Intuitionistic Logic: Philosophical Motivation

Intuitionists’ view-point on the meaning of the basic logical and set theoretical concepts used in mathematics is different from that of most mathematicians use in their research.

The basic difference between the intuitionist and classical mathematician lies in the interpretation of the word exists. For example, let $A(x)$ be a statement in the arithmetic of natural numbers. For the mathematicians the sentence $\exists x A(x)$ is true if it is a theorem of arithmetic.

If a mathematician proves sentence $\exists x A(x)$ this does not always mean that he is able to indicate a method of construction of a natural number $n$ such that $A(n)$ holds.
Moreover, the mathematician often obtains the proof of the existential sentence $\exists x A(x)$ by proving first a sentence

$$\neg \forall x \neg A(x)$$

Next he makes use of a classical tautology

$$(\neg \forall x \neg A(x)) \Rightarrow \exists x A(x))$$

By applying Modus Ponens he obtains the proof of the existential sentence

$$\exists x A(x)$$

For the intuitionist such method is not acceptable, for it does not give any method of constructing a number $n$ such that $A(n)$ holds
Intuitionistic Logic: Philosophical Motivation

For this reason the intuitionist do not accept the classical tautology

\[(\neg \forall x \neg A(x)) \Rightarrow \exists x A(x)\]

as intuitionistic tautology, or as an intuitionistically provable sentence
Intuitionistic Logic: Philosophical Motivation

Let us denote by ⊢_I A and ⊨_I A the fact that A is intuitionistically provable and that A is intuitionistic tautology, respectively.

The proof system I for the intuitionistic logic has hence to be such that

\[ \forall I \ (\neg \forall x \neg A(x)) \Rightarrow \exists x A(x) \]

The intuitionistic semantics I has to be such that

\[ \not \models_I (\neg \forall x \neg A(x)) \Rightarrow \exists x A(x) \]
Intuitionistic Logic: Philosophical Motivation

The above means also that intuitionists interpret differently the meaning of propositional connectives.

Intuitionistic implication
The intuitionistic implication \((A \Rightarrow B)\) is considered by to be true if there exists a method by which a proof of \(B\) can be deduced from the proof of \(A\).
In the case of the implication

\[
i(\neg\forall x \neg A(x)) \Rightarrow \exists x A(x)
\]

there is no general method which, from a proof of the sentence

\[
(\neg\forall x \neg A(x))
\]
permits us to obtain an intuitionistic proof of the sentence

\[
\exists x A(x)
\]
Intuitionistic Logic: Philosophical Motivation

**Intuitionistic negation**
The sentence $\neg A$ is considered **intuitionistically true** if the acceptance of the sentence $A$ leads to **absurdity**.

As a result of above understanding of negation and implication we have that in the intuitionistic proof system $I$

$$\vdash_I (A \Rightarrow \neg \neg A) \quad \text{but} \quad \nvDash_I (\neg \neg A \Rightarrow A)$$

Consequently, the intuitionistic semantics $I$ has to be such that

$$\models_I (A \Rightarrow \neg \neg A) \quad \text{and} \quad \nvDash_I (\neg \neg A \Rightarrow A)$$
Intuitionistic Logic: Philosophical Motivation

Intuitionistic disjunction
The intuitionist regards a disjunction \((A \cup B)\) as true if one of the sentences \(A, B\) is true and there is a method by which it is possible to find out which of them is true.

As a consequence a classical law of excluded middle
\[(A \cup \neg A)\]
is not acceptable by the intuitionists.

This means that the intuitionistic proof system \(I\) must be such that
\[\not \vdash_I (A \cup \neg A)\]
and the intuitionistic semantics \(I\) has to be such that
\[\not \models_I (A \cup \neg A)\]
Chapter 7
Introduction to Intuitionistic and Modal Logics

PART 2: Intuitionistic Proof System $\mathcal{PI}$,
Algebraic Semantics and Completeness Theorem
We define now a Hilbert style proof system with a set of axioms that is due to Rasiowa (1959). We adopted this axiomatization for two reasons.

First reason is that it is the most natural and appropriate set of axioms to carry the algebraic proof of the completeness theorem.

Second reason is that they clearly describe the main difference between intuitionistic and classical logic. Namely, by adding to \( I \) the only one more axiom

\[(A \cup \neg A)\]

we get a complete formalization for classical logic.
Intuitionistic Proof System I

Here are the components if the proof system I

Language
We adopt a propositional language

\[ L = L_{\{\cup, \cap, \Rightarrow, \neg\}} \]

with the set of formulas \( F \)

Axioms
A1 \((A \Rightarrow B) \Rightarrow ((B \Rightarrow C) \Rightarrow (A \Rightarrow C)))\)
A2 \((A \Rightarrow (A \cup B))\)
A3 \((B \Rightarrow (A \cup B))\)
A4 \(((A \Rightarrow C) \Rightarrow ((B \Rightarrow C) \Rightarrow ((A \cup B) \Rightarrow C)))\)
A5 \(((A \cap B) \Rightarrow A)\)
A6 \(((A \cap B) \Rightarrow B)\)
A7 \(((C \Rightarrow A) \Rightarrow ((C \Rightarrow B) \Rightarrow (C \Rightarrow (A \cap B))))\)
Intuitionistic Proof System I

A7 \(((C \Rightarrow A) \Rightarrow ((C \Rightarrow B) \Rightarrow (C \Rightarrow (A \cap B))))\)
A8 \(((A \Rightarrow (B \Rightarrow C)) \Rightarrow ((A \cap B) \Rightarrow C))\)
A9 \(((A \cap B) \Rightarrow C) \Rightarrow (A \Rightarrow (B \Rightarrow C))\),
A10 \((A \cap \neg A) \Rightarrow B\),
A11 \((A \Rightarrow (A \cap \neg A)) \Rightarrow \neg A\),

where \(A, B, C\) are any formulas in \(\mathcal{L}\)

Rules of inference
We adopt the Modus Ponens

\[
\frac{A ; (A \Rightarrow B)}{(MP) \ B}
\]

as the only rule of inference
Intuitionistic Proof System $I$

A proof system

$I = ( \mathcal{L}, \mathcal{F} A_1 - A_{11}, (MP) )$

for axioms $A_1 - A_{11}$ defined above is called a Hilbert style formalization for intuitionistic propositional logic.

We introduce, as usual, the notion of a formal proof in $I$ and denote by

$\vdash_I A$

the fact that a formula $A$ has a formal proof in $I$ or that $A$ is provable in $I$. 
Algebraic Semantics and Completeness Theorem
Algebraic Semantics

We present now a short version of Tarski, Rasiowa, and Sikorski pseudo-Boolean algebra semantics.

We also discuss the algebraic completeness theorem for the intuitionistic propositional logic.

We leave the Kripke semantics for the reader to explore from other, multiple sources.
Here are some basic definitions

Relatively Pseudo-Complemented Lattice (Birkhoff, 1935)
A lattice \((B, \cap, \cup)\)

is said to be relatively pseudo-complemented if and only if for any elements \(a, b \in B\), there exists the greatest element \(c\), such that

\[ a \cap c \leq b \]

Such greatest element \(c\) is denoted by \(a \Rightarrow b\) and called the pseudo-complement of a relative to \(b\)
Algebraic Semantics

Directly from definition we have that

\[ (*) \quad x \leq a \Rightarrow b \quad \text{if and only if} \quad a \cap x \leq b \quad \text{for all} \quad x, a, b \in B \]

This equation (\( (*) \)) can serve as the definition of the relative pseudo-complement \( a \Rightarrow b \).
Algebraic Semantics

Fact
Every relatively pseudo-complemented lattice \((B, \cap, \cup)\) has the greatest element, called a unit element and denoted by 1.

Proof
Observe that \(a \cap x \leq a\) for all \(x, a \in B\).
By (*) we have that \(x \leq a \Rightarrow a\) for all \(x \in B\).
This means that \(a \Rightarrow a\) is the greatest element in the lattice \((B, \cap, \cup)\). We write it as
\[
a \Rightarrow a = 1
\]
Definition

An abstract algebra

\[ \mathcal{B} = (B, 1, \Rightarrow, \cap, \cup) \]

is said to be a relatively pseudo-complemented lattice if and only if \((B, \cap, \cup)\) is a relatively pseudo-complemented lattice with the relative pseudo-complement \(\Rightarrow\) defined by the equation

\[ \text{(*) } x \leq a \Rightarrow b \quad \text{if and only if} \quad a \cap x \leq b \quad \text{for all } x, a, b \in B \]

and with the unit element 1.
Algebraic Semantics

Relatively Pseudo-complemented Set Lattices

Consider a topological space $X$ with an interior operation $I$.

Let $G(X)$ be the class of all open subsets of $X$ and $G^*(X)$ be the class of all both dense and open subsets of $X$.

Then the algebras

$$(G(X), X, \cup, \cap, \Rightarrow), \quad (G^*(X), X, \cup, \cap, \Rightarrow)$$

where $\cup, \cap$ are set-theoretical operations of union, intersection, and $\Rightarrow$ is defined by

$$Y \Rightarrow Z = I(X - Y) \cup Z$$

are relatively pseudo-complemented lattices.
Algebraic Semantics

Clearly, all sub algebras of these algebras are also relatively pseudo-complemented lattices.

They are typical examples of relatively pseudo-complemented lattices.
Algebraic Semantics

Pseudo - Boolean Algebra (Heyting Algebra)

An algebra

\[ B = (B, 1, 0, \Rightarrow, \cap, \cup, \neg) \]

is said to be a pseudo - Boolean algebra if and only if

\[ (B, 1, \Rightarrow, \cap, \cup) \]

is a relatively pseudo-complemented lattice in which a zero element 0 exists and \( \neg \) is a one argument operation defined as follows

\[ \neg a = a \Rightarrow 0 \]
Algebraic Semantics

The operation \( \neg \) defined as

\[ \neg a = a \Rightarrow 0 \]

is called a **pseudo-complementation**

The **pseudo-Boolean** algebras are also called **Heyting** algebras to stress their connection to the **intuitionistic** logic
Algebraic Semantics

Let $X$ be **topological** space with an interior operation $I$. Let $G(X)$ be the class of all open subsets of $X$. Then

$$(G(X), X, \emptyset, \cup, \cap, \Rightarrow, \neg)$$

where $\cup, \cap$ are set-theoretical operations of union, intersection, and $\Rightarrow$ is defined by

$$Y \Rightarrow Z = I(X - Y) \cup Z$$

and $\neg$ is defined as

$$\neg Y = Y \Rightarrow \emptyset = I(X - Y), \quad \text{for all } Y \subseteq X$$

is a **pseudo - Boolean** algebra.

Every sub algebra of $G(X)$ is also a pseudo-Boolean algebra. They are called **pseudo-fields of sets**.
Algebraic Semantics

The following theorem states that pseudo-fields are typical examples of pseudo-Boolean algebras.

The theorems of this type are often called Stone Representation Theorems to remember an American mathematician H. M. Stone.

Stone was one of the first to initiate the investigations of relationship between logic and general topology in the article „The Theory of Representations for Boolean Algebras“, Trans. of the Amer. Math. Soc 40, 1936.
Representation Theorem \textit{(McKinsey, Tarski, 1946)}

For every \textit{pseudo-Boolean} algebra $\mathcal{B} = (B, 1, 0, \Rightarrow, \cap, \cup, \neg)$

there exists a \textit{monomorphism} $h$ of $\mathcal{B}$ into a \textit{pseudo-field} $G(X)$ of all \textit{open} subsets of a \textit{compact} topological $T_0$ space $X$
Intuitionistic Algebraic Model

We say that a formula \( A \) is an **intuitionistic tautology** if and only if

any **pseudo-Boolean** algebra \( B \) is a **model** for \( A \)

This kind of **models** because their **connection** to abstract algebras are called **algebraic models**

We put it formally as follows.
Intuitionistic Algebraic Model

Let $A$ be a formula of the language $L_{\{\cup, \cap, \Rightarrow, \neg\}}$ and let

$$B = (B, 1, 0, \Rightarrow, \cap, \cup, \neg)$$

be a \textbf{pseudo - Boolean} algebra

We say that the algebra $B$ is a \textbf{model} for the formula $A$ and denote it by

$$B \models A$$

if and only if $v^*(A) = 1$ holds for all variables assignments

$$v : \text{VAR} \rightarrow B$$
Intuitionistic Tautology

The formula $A$ is an intuitionistic tautology and is denoted by

$\models I_A$

if and only if

$B \models A$ for all pseudo-Boolean algebras $B$
Intuitionistic Tautology

In **Algebraic Logic** the notion of **tautology** is often defined using a notion

"a formula $A$ is **valid** in an algebra $B$"

It is formally defined as follows
Intuitionistic Tautology

Definition
A formula \( A \) is valid in a pseudo-Boolean algebra

\[ \mathcal{B} = (B, 1, 0, \Rightarrow, \cap, \cup, \neg) \]

if and only if \( v^*(A) = 1 \) holds for all variables assignments \( v : \text{VAR} \rightarrow B \)

Directly from definitions we get the following Fact
Intuitionistic Tautology

**Fact**

For any formula $A$, $\models A$ if and only if $A$ is valid in all pseudo-Boolean algebras $\mathcal{B}$.

The **Fact** is often used as an equivalent definition of the intuitionistic tautology.
We write now \( \vdash I A \) to denote any proof system for the intuitionistic propositional logic, and in particular the Rasiowa (1959) proof system we have defined.

**Intuitionistic Completeness Theorem** (Mostowski 1948)

For any formula \( A \) of \( \mathcal{L}_{\{\cup, \cap, \Rightarrow, \neg\}} \),

\[ \vdash I A \text{ if and only if } \models I A \]

The intuitionistic completeness theorem follows directly from the general **algebraic completeness theorem** that combines results of Mostowski (1958), Rasiowa (1951) and Rasiowa-Sikorski (1957).
Algebraic Completeness

Algebraic Completeness Theorem
For any formula \( A \) the following conditions are equivalent

(i) \( \vdash A \)

(ii) \( \models A \)

(iii) \( A \) is valid in every pseudo-Boolean algebra

\[
(\mathcal{G}(X), X, \emptyset, \cup, \cap, \Rightarrow, \neg)
\]

of open subsets of any topological space \( X \)

(iv) \( A \) is valid in every pseudo-Boolean algebra \( \mathcal{B} \) with at most \( 2^{2^r} \) elements, where \( r \) is the number of all subformulas of \( A \)

Moreover, each of the conditions (i) - (iv) is equivalent to the following one.

(v) \( A \) is valid in the pseudo-Boolean algebra

\[
(\mathcal{G}(X), X, \emptyset, \cup, \cap, \Rightarrow, \neg)
\]

of open subsets of a dense-in-itself metric space \( X \neq \emptyset \) (in particular of an \( n \)-dimensional Euclidean space \( X \))
Chapter 7
Introduction to Intuitionistic and Modal Logics

PART 3: Intuitionistic Tautologies and Connection with Classical Tautologies
Intuitionistic Tautologies

Here are some important basic classical tautologies that are also intuitionistic tautologies

\[(A \Rightarrow A)\]
\[(A \Rightarrow (B \Rightarrow A))\]
\[(A \Rightarrow (B \Rightarrow (A \cap B)))\]
\[((A \Rightarrow (B \Rightarrow C)) \Rightarrow ((A \Rightarrow B) \Rightarrow (A \Rightarrow C)))\]
\[(A \Rightarrow \neg \neg A)\]
\[\neg (A \cap \neg A)\]
\[((\neg A \cup B) \Rightarrow (A \Rightarrow B))\]

Of course, all of logical axioms A1 - A11 of the proof system I are also classical and intuitionistic tautologies
Intuitionistic Tautologies

Here are some more of important classical tautologies that are intuitionistic tautologies

\[ (\neg A \cup B) \Rightarrow (A \Rightarrow B) \]
\[ (\neg (A \cup B) \Rightarrow (\neg A \cap \neg B)) \]
\[ (\neg A \cap \neg B) \Rightarrow (\neg (A \cup B)) \]
\[ (\neg A \cup \neg B) \Rightarrow \neg (A \cap B) \]
\[ (A \Rightarrow B) \Rightarrow (\neg B \Rightarrow \neg A) \]
\[ (A \Rightarrow \neg B) \Rightarrow (B \Rightarrow \neg A) \]
\[ (\neg \neg \neg A \Rightarrow \neg A) \]
\[ (\neg A \Rightarrow \neg \neg \neg A) \]
\[ (\neg \neg (A \Rightarrow B) \Rightarrow (A \Rightarrow \neg \neg B)) \]
\[ (C \Rightarrow A) \Rightarrow ((C \Rightarrow (A \Rightarrow B)) \Rightarrow (C \Rightarrow B)) \]
Intuitionistic Tautologies

Here are some important classical tautologies that are not intuitionistic tautologies

\[(A \cup \neg A)\]
\[(\neg\neg A \Rightarrow A)\]
\[((A \Rightarrow B) \Rightarrow (\neg A \cup B))\]
\[(\neg (A \cap B) \Rightarrow (\neg A \cup \neg B))\]
\[((\neg A \Rightarrow B) \Rightarrow (\neg B \Rightarrow A))\]
\[((\neg A \Rightarrow \neg B) \Rightarrow (B \Rightarrow A))\]
\[((A \Rightarrow B) \Rightarrow A) \Rightarrow A)\]
Connection Between Classical and Intuitionistic Logics
Connection Between Classical and Intuitionistic Logics

The first connection is quite obvious. It was proved by Rasiowa, Sikorski in 1964 that by adding the axiom

\[ A12 \ (A \cup \neg A) \]

to the set of logical axioms A1 - A11 of the proof system I, we obtain a proof system C that is complete with respect to classical semantics.

This proves the following

**Theorem 1**

Every formula that is intuitionistically derivable is also classically derivable, i.e. the implication

\[ \vdash_A \text{ then } \vdash_C A \]

holds for any \( A \in \mathcal{F} \)
We write $\models A$ and $\models_I A$ to denote that $A$ is a classical and intuitionistic tautology, respectively.

As both proof systems $I$ and $C$ are complete under respective semantics, we can re-write Theorem 1 as the following relationship between classical and intuitionistic tautologies.

**Theorem 2**

For any formula $A \in \mathcal{F}$,

If $\models_I A$, then $\models A$
Classical and Intuitionistic Logics

The next relationship shows how to obtain intuitionistic tautologies from the classical tautologies and vice versa. The following has been proved by Glivenko in 1929 and independently by Tarski in 1938:

**Theorem 3 (Glivenko, Tarski)**
For any formula $A \in F$, $A$ is classically provable if and only if $\neg\neg A$ is intuitionistically provable, i.e.

$$\vdash A \quad \text{if and only if} \quad \vdash_I \neg\neg A$$

where we use symbol $\vdash$ for classical provability.
Classical and Intuitionistic Logics

Theorem 4 (Tarski, 1938)
For any formula $A \in \mathcal{F}$,
$A$ is a classical tautology if and only if $\neg\neg A$ is an intuitionistic tautology, i.e.

$$\models A \text{ if and only if } \models I \neg\neg A$$
Theorem 5 (Gödel, 1931)

For any formulas $A, B \in \mathcal{F}$, a formula $(A \Rightarrow \neg B)$ is classically provable if and only if it is intuitionistically provable, i.e.

$\vdash (A \Rightarrow \neg B)$ if and only if $\vdash I (A \Rightarrow \neg B)$
Classical and Intuitionistic Logics

Theorem 6 (Gödel, 1931)
For any formula $A, B \in \mathcal{F}$, if $A$ contains no connectives except $\cap$ and $\neg$, then $A$ is classically provable if and only if it is intuitionistically provable, i.e

$$\vdash A \quad \text{if and only if} \quad \vdash_I A$$
Classical and Intuitionistic Logics

By the completeness of classical and intuitionistic logics we get the following semantic version of Gödel’s Theorems 5, 6

**Theorem 7**

A formula \( (A \Rightarrow \neg B) \) is a classical tautology if and only if it is an intuitionistic tautology, i.e.

\[
\models (A \Rightarrow \neg B) \quad \text{if and only if} \quad \models_I (A \Rightarrow \neg B)
\]

**Theorem 8**

If a formula \( A \) contains no connectives except \( \cap \) and \( \neg \), then

\[
\models A \quad \text{if and only if} \quad \models_I A
\]
On intuitionistically derivable disjunction

In classical logic it is possible for the disjunction

\[(A \cup B)\]

to be a tautology when neither \(A\) nor \(B\) is a tautology

The tautology \((A \cup \neg A)\) is the simplest example

This does not hold for the intuitionistic logic

This fact was stated without the proof by Gödel in 1931 and proved by Gentzen in 1935 via his proof system LI which was discussed shortly in chapter 6 and is covered in detail in this chapter and the next set of slides.
On intuitionistically derivable disjunction

The following theorem was announced without proof by Gödel in 1931 and proved by Gentzen in 1935

**Theorem 9 (Gödel, Gentzen)**

A disjunction \((A \cup B)\) is intuitionistically provable if and only if either \(A\) or \(B\) is intuitionistically provable i.e.

\[\vdash_I (A \cup B) \text{ if and only if } \vdash_I A \text{ or } \vdash_I B\]
On intuitionistically derivable disjunction

We obtain, via the Completeness Theorems the following semantic version of the above (Gödel, Gentzen) Theorem 9.

**Theorem 10**

A disjunction \((A \cup B)\) is intuitionistic tautology if and only if either \(A\) or \(B\) is intuitionistic tautology, i.e.

\[ \models I(A \cup B) \text{ if and only if } \models I A \text{ or } \models I B \]