cse371/math371
LOGIC

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LECTURE 4a

Chapter 4 Review

PART 1: DEFINITIONS

PART 2: Problems

PART 1: Definitions from Chapter 4 you have to know

Definition: Proof System

Definition 1

By a **proof system** we understand a quadruple

$$S = (\mathcal{L}, \mathcal{E}, LA, \mathcal{R})$$

where

$\mathcal{L} = \{\mathcal{A}, \mathcal{F}\}$ is a **language** of S with a set $\mathcal{F}$ of formulas

$\mathcal{E}$ is a set of **expressions** of S

In particular case $\mathcal{E} = \mathcal{F}$

$LA \subseteq \mathcal{E}$ is a **non-empty, finite** set of logical axioms of S

$\mathcal{R}$ is a **non-empty, finite set** of rules of inference of S

Definition: Sound Rule of Inference

Definition 2

An inference rule

$$(r) \quad \frac{P_1 \;\; ; \;\; P_2 \;\; ; \;\; .... \;\; ; \;\; P_m}{C}$$

is sound under a semantics **M** if and only if all **M** - models of the set $\{P_1, P_2, \dots P_m\}$ of its **premisses** are also **M** - models of its **conclusion** C

In particular, in case of **extensional propositional semantics** when the condition below holds for any truth assignment $v : VAR \longrightarrow LV$

If $v \models_{\mathbf{M}} \{P_1, P_2, \dots P_m\}$, **then** $v \models_{\mathbf{M}} C$

Definition: Direct Consequence

Definition 3

For any rule of inference $r \in \mathcal{R}$ of the form

$$(r) \quad \frac{P_1 \quad ; \quad P_2 \quad ; \quad .... \quad ; \quad P_m}{C}$$

C is called a **direct consequence** of $P_1, \dots P_m$ by virtue of the rule $r \in \mathcal{R}$

Definition: Formal Proof

Definition 4

A formal proof of an expression $E \in \mathcal{E}$ in a proof system $S = (\mathcal{L}, \mathcal{E}, LA, \mathcal{R})$ is a sequence

$$A_1, A_2, \ldots, A_n \quad \text{for} \quad n \geq 1$$

of expressions from $\mathcal{E}$, such that

$$A_1 \in LA, \qquad A_n = E$$

and for each $1 < i \leq n$, either $A_i \in LA$ or A_i is a **direct consequence** of some of the **preceding expressions** by virtue of **one** of the rules of inference

$n \geq 1$ is the **length** of the proof $A_1, A_2, \ldots, A_n$

NOTATION: Provable Expressions

Notation

We write $\vdash_S E$ to denote that $E \in \mathcal{E}$ **has a formal proof** in the proof system S

A set

$$\mathbf{P}_S = \{E \in \mathcal{E} : \ \vdash_S E\}$$

is called the set of **all provable expressions** in S

Definition: Sound S

Definition 5

Given a proof system

$$S = (\mathcal{L}, \mathcal{E}, LA, \mathcal{R})$$

We say that the system S is **sound** under a semantics **M** iff the following conditions hold

1. Logical axioms LA are **tautologies** of under the semantics **M**, i.e.

$$LA \subseteq \mathbf{T_M}$$

2. Each **rule of inference** $r \in \mathcal{R}$ is **sound** under the semantics **M**

THEOREMS: Soundness Theorem

Let $\mathbf{P}_S$ be the set of all provable expressions of S i.e.

$$\mathbf{P}_S = \{A \in \mathcal{E} : \ \vdash_S A\}$$

Let $\mathbf{T}_{\mathbf{M}}$ be a set of all expressions of S that are **tautologies** under a semantics **M**, i.e.

$$\mathbf{T}_{\mathbf{M}} = \{A \in \mathcal{E} : \ \models_{\mathbf{M}} A\}$$

Our GOAL is to prove the following theorems:

Soundness Theorem (for S and semantics **M**)

$$\mathbf{P}_S \subseteq \mathbf{T}_{\mathbf{M}}$$

i.e. for any $A \in \mathcal{E}$, the following implication holds

If $\vdash_S A$ **then** $\models_{\mathbf{M}} A$

THEOREMS: Completeness Theorem

Completeness Theorem (for S and semantics **M**)

$$\mathbf{P}_S = \mathbf{T}_\mathbf{M}$$

i.e. for any $A \in \mathcal{E}$, the following holds

$$\vdash_S A \quad \text{if and only if} \quad \models_\mathbf{M} A$$

The **Completeness Theorem** consists of two parts:

Part 1: **Soundness Theorem**

$$\mathbf{P}_S \subseteq \mathbf{T}_\mathbf{M}$$

Part 2: **Completeness Part** of the Completeness Theorem

$$\mathbf{T}_\mathbf{M} \subseteq \mathbf{P}_S$$

PART 2: Simple Problems

Problem 1

Given a proof system:

$$S = (\mathcal{L}_{\{\neg,\Rightarrow\}},\ \mathcal{F},\ \ \{(A \Rightarrow A), (A \Rightarrow (\neg A \Rightarrow B))\},\ \mathcal{R} = \{(r)\}$$

where $(r)\ \dfrac{(A \Rightarrow B)}{(B \Rightarrow (A \Rightarrow B))})$

Write a **formal proof** in S with 2 applications of the rule (r)

Solution: There are many solutions. Here is one of them.

Required formal proof is a sequence $A_1, A_2, A_3,$ where

$A_1 = (A \Rightarrow A)$
(Axiom)

$A_2 = (A \Rightarrow (A \Rightarrow A))$
Rule (r) application 1 for $A = A,\ B = A$

$A_3 = ((A \Rightarrow A) \Rightarrow (A \Rightarrow (A \Rightarrow A)))$
Rule (r) application 2 for $A = A, B = (A \Rightarrow A)$

Given a proof system:

$$S = (\mathcal{L}_{\{\neg,\Rightarrow\}}, \mathcal{F},\ \{(A \Rightarrow A), (A \Rightarrow (\neg A \Rightarrow B))\}, (r)\ \frac{(A \Rightarrow B)}{(B \Rightarrow (A \Rightarrow B))})$$

Problem 2

Prove that S is **sound** under classical semantics.

Solution

1. Both axioms of S are basic classical tautologies

2. Consider the rule of inference of S

$$(r)\ \frac{(A \Rightarrow B)}{(B \Rightarrow (A \Rightarrow B))}$$

Assume that its premise (the only premise) is true, i.e. let v be any truth assignment, such that $v^*(A \Rightarrow B) = T$

We evaluate logical value of the conclusion under the truth assignment v as follows

$$v^*(B \Rightarrow (A \Rightarrow B)) = v^*(B) \Rightarrow T = T$$

for any B and any value of $v^*(B)$

Given a proof system:

$$S = (\mathcal{L}_{\{\neg,\Rightarrow\}}, \mathcal{F},\ \{(A \Rightarrow A), (A \Rightarrow (\neg A \Rightarrow B))\}, (r)\ \frac{(A \Rightarrow B)}{(B \Rightarrow (A \Rightarrow B))})$$

Problem 3.

Write a **formal proof** of your choice in S with 2 applications of the rule (r)

Solution

There many of such proofs, of different length, with different choice if axioms - here is my choice: A_1, A_2, A_3, where

$A_1 = (A \Rightarrow A)$

(Axiom)

$A_2 = (A \Rightarrow (A \Rightarrow A))$

Rule (r) application 1 for $A = A,\ B = A$

$A_3 = ((A \Rightarrow A) \Rightarrow (A \Rightarrow (A \Rightarrow A)))$

Rule (r) application 2 for $A = A, B = (A \Rightarrow A)$

Formal Proof

Given a proof system:

$$S = (\mathcal{L}_{\{\neg, \Rightarrow\}}, \mathcal{F},\ \{(A \Rightarrow A), (A \Rightarrow (\neg A \Rightarrow B))\}, (r)\ \frac{(A \Rightarrow B)}{(B \Rightarrow (A \Rightarrow B))})$$

Problem 4

1. Prove, by constructing a **formal proof** that

$$\vdash_S ((\neg A \Rightarrow B) \Rightarrow (A \Rightarrow (\neg A \Rightarrow B)))$$

Solution Required formal proof is a sequence A_1, A_2, where

$A_1 = (A \Rightarrow (\neg A \Rightarrow B))$

Axiom

$A_2 = ((\neg A \Rightarrow B) \Rightarrow (A \Rightarrow (\neg A \Rightarrow B)))$

Rule (r) application for $A = A, B = (\neg A \Rightarrow B)$

Soundness Theorem

2. Does above point **1.** prove that

$$\models ((\neg A \Rightarrow B) \Rightarrow (A \Rightarrow (\neg A \Rightarrow B)))?$$

Solution

Yes, it does because the system S is **sound** and we proved by Mathematical Induction over the length of a proof that if S is **sound**, then the **Soundness Theorem** holds for S

Soundness

Problem 5

Given a proof system:

$$S = (\mathcal{L}_{\{\neg,\Rightarrow\}},\ \mathcal{F},\ \{(A \Rightarrow A), (A \Rightarrow (\neg A \Rightarrow B))\},\ (r)\ \frac{(A\Rightarrow B)}{(B\Rightarrow(A\Rightarrow B))})$$

Prove that S is **not sound** under **K** semantics

Solution

Axiom $(A \Rightarrow A)$ is not a **K** semantics tautology

Any truth assignment v such that $v^*(A) = \perp$ is a **counter-model** for it

This proves that S is **not sound** under **K** semantics

Soundness

Given a proof system

$$S = (\ \mathcal{L}_{\{\Rightarrow,\cup,\neg\}},\ \ \mathcal{F},\ \ \mathbf{A1},\ \ (r1),\ (r2)\ )$$

A1 $(A \Rightarrow (A \cup B))$, for any $A, B \in \mathcal{F}$

Rules of inference

$$(r1)\quad \frac{A\ ;B}{(A \cup \neg B)}, \qquad\qquad (r2)\quad \frac{A\ ;(A \cup B)}{B},$$

for any formulas $A, B \in \mathcal{F}$

1. Verify whether S is **sound/not sound** under **classical** semantics.

Soundness

Rules of inference

$$(r1) \quad \frac{A \; ; B}{(A \cup \neg B)}, \qquad (r2) \quad \frac{A \; ; (A \cup B)}{B}$$

The Logical Axiom **A1** is a **basic tautology**

The rule $(r1)$ is **sound** because for any v,

if $v^*(A) = v^*(B) = T$, then $v^*((A \cup \neg B)) = T \cup F = T$

The rule $(r2)$ is **not sound**

Take any v such that it evaluates $A = T$ and $B = F$

The premiss $(A \cup B$ of the rule $(r2)$ is $T \cup F = T$ and the conclusion B is F

This proves that the system S is **not sound**

Formal Proof

Rules of inference

$$(r1)\quad \frac{A\ ;\ B}{(A \cup \neg B)}, \qquad\qquad (r2)\quad \frac{A\ ;\ (A \cup B)}{B}$$

Write down a **formal proof** of the formula

$$\neg(a \Rightarrow (a \cup b))$$

in the **proof system** S, i.e. write all components

$$B_1,\ B_2,\ \ldots\ B_n$$

of the proof with **comments** how they were obtained

Formal Proof

Rules of inference

$(r1)\quad \frac{A\ ;B}{(A\cup\neg B)},\qquad (r2)\quad \frac{A\ ;(A\cup B)}{B}$

Axiom $(A \Rightarrow (A \cup B))$

Formal Proof of $\neg(a \Rightarrow (a \cup b))$ is:

B_1: $(a \Rightarrow (a \cup b))$ Axiom or $A = a,\ B = b$

B_2: $(a \Rightarrow (a \cup b))$ Axiom for $A = a,\ B = b$

B_3: $((a \Rightarrow (a \cup b)) \cup \neg(a \Rightarrow (a \cup b)))$

rule (r1) application to B_1 and B_2 for
$A = (a \Rightarrow (a \cup b)),\ \ B = (a \Rightarrow (a \cup b))$

B_4: $\neg(a \Rightarrow (a \cup b))$ rule (r2) application to B_1 and B_3 for
$A = (a \Rightarrow (a \cup b)),\ \ B = \neg(a \Rightarrow (a \cup b))$

Soundness

We proved that

$$\vdash_S \neg(a \Rightarrow (a \cup b))$$

Does the above prove that

$$\models \neg(a \Rightarrow (a \cup b))?$$

No, it doesn't

We proved that the proof system S is **not sound**, so the existence of a proof does not guarantee that what we proved is a tautology.

Moreover, the proof of $\neg(a \Rightarrow (a \cup b))$ used rule $(r2)$ that is **not sound**

STRONG Soundness

Given a proof system

$$S = (\ \mathcal{L}_{\{\Rightarrow,\cup,\neg\}},\ \ \mathcal{F},\ \ \mathbf{A1},\ \ (r1),\ (r2)\ )$$

A1 $(A \Rightarrow (A \cup B))$, for any $A, B \in \mathcal{F}$

Rules of inference

$$(r1)\ \ \frac{\neg A\ ;\ B}{(A \Rightarrow B)}, \qquad (r2)\ \ \frac{A\ ;\ \neg B}{\neg(A \Rightarrow B)},$$

for any formulas $A, B \in \mathcal{F}$

1. Prove that S is **sound** but not **strongly sound** under classical semantics.

2. Prove that $\mathbf{P}_S = \{(A \Rightarrow (A \cup B)) : \quad A, B \in \mathcal{F}.\}$