cse371/math371
LOGIC

Professor Anita Wasilewska
LECTURE 4a
Chapter 4 Review

PART 1: DEFINITIONS
PART 2: Problems
PART 1: Definitions from Chapter 4 you have to know
Definition: Proof System

Definition 1

By a **proof system** we understand a quadruple

\[ S = (\mathcal{L}, \mathcal{E}, \mathcal{LA}, \mathcal{R}) \]

where

- \( \mathcal{L} = \{\mathcal{A}, \mathcal{F}\} \) is a **language** of \( S \) with a set \( \mathcal{F} \) of formulas
- \( \mathcal{E} \) is a set of **expressions** of \( S \)
- In particular case \( \mathcal{E} = \mathcal{F} \)
- \( \mathcal{LA} \subseteq \mathcal{E} \) is a **non-empty, finite** set of **logical axioms** of \( S \)
- \( \mathcal{R} \) is a **non-empty, finite set** of **rules of inference** of \( S \)
Definition: Sound Rule of Inference

Definition 2

An inference rule

\[(r) \quad \frac{P_1 ; P_2 ; \ldots ; P_m}{C}\]

is sound under a semantics \(M\) if and only if all \(M\)-models of the set \(\{P_1, P_2, \ldots, P_m\}\) of its premises are also \(M\)-models of its conclusion \(C\).

In particular, in case of extensional propositional semantics when the condition below holds for any truth assignment \(v: VAR \rightarrow LV\):

If \(v \models_M \{P_1, P_2, \ldots, P_m\}\), then \(v \models_M C\)
Definition: Direct Consequence

Definition 3
For any rule of inference \( r \in \mathcal{R} \) of the form

\[
(r) \quad P_1 ; P_2 ; \ldots ; P_m \quad \frac{}{C}
\]

\( C \) is called a **direct consequence** of \( P_1, \ldots, P_m \) by virtue of the rule \( r \in \mathcal{R} \).
Definition: Formal Proof

Definition 4
A formal proof of an expression $E \in \mathcal{E}$ in a proof system $S = (\mathcal{L}, \mathcal{E}, LA, R)$ is a sequence

$$A_1, A_2, \ldots, A_n \text{ for } n \geq 1$$

of expressions from $\mathcal{E}$, such that

$$A_1 \in LA, \quad A_n = E$$

and for each $1 < i \leq n$, either $A_i \in LA$ or $A_i$ is a direct consequence of some of the preceding expressions by virtue of one of the rules of inference.

$n \geq 1$ is the length of the proof $A_1, A_2, \ldots, A_n$.
NOTATION: Provable Expressions

Notation

We write $\vdash_S E$ to denote that $E \in \mathcal{E}$ has a formal proof in the proof system $S$.

A set

$$P_S = \{E \in \mathcal{E} : \vdash_S E\}$$

is called the set of all provable expressions in $S$. 
Definition: Sound $S$

Definition 5
Given a proof system

$$S = (\mathcal{L}, \mathcal{E}, LA, R)$$

We say that the system $S$ is **sound** under a semantics $M$ iff the following conditions hold

1. Logical axioms $LA$ are **tautologies** of under the semantics $M$, i.e.

$$LA \subseteq T_M$$

2. Each **rule of inference** $r \in R$ is **sound** under the semantics $M$
THEOREMS: Soundness Theorem

Let \( P_S \) be the set of all provable expressions of \( S \) i.e.

\[
P_S = \{ A \in \mathcal{E} : \vdash_S A \}\]

Let \( T_M \) be a set of all expressions of \( S \) that are tautologies under a semantics \( M \), i.e.

\[
T_M = \{ A \in \mathcal{E} : \models_M A \}\]

Our GOAL is to prove the following theorems:

**Soundness Theorem** (for \( S \) and semantics \( M \))

\[
P_S \subseteq T_M
\]

i.e. for any \( A \in \mathcal{E} \), the following implication holds

If \( \vdash_S A \) then \( \models_M A \)
Completeness Theorem (for $S$ and semantics $M$)

$$P_S = T_M$$

i.e. for any $A \in \mathcal{E}$, the following holds

$$\vdash_S A \quad \text{if and only if} \quad \models_M A$$

The **Completeness Theorem** consists of two parts:

**Part 1:**  **Soundness Theorem**

$$P_S \subseteq T_M$$

**Part 2:**  **Completeness Part** of the Completeness Theorem

$$T_M \subseteq P_S$$
PART 2: Simple Problems
Problem 1
Given a proof system:

\[ S = (\mathcal{L}_{\neg,\Rightarrow}, \mathcal{F}, \{(A \Rightarrow A), (A \Rightarrow (\neg A \Rightarrow B))\}), \mathcal{R} = \{(r)\} \]

where \((r) \frac{(A \Rightarrow B)}{(B \Rightarrow (A \Rightarrow B))}\)

Write a **formal proof** in \( S \) with 2 applications of the rule \((r)\)

**Solution:** There are many solutions. Here is one of them.

Required formal proof is a sequence \( A_1, A_2, A_3 \), where

\( A_1 = (A \Rightarrow A) \)  
(Axiom)

\( A_2 = (A \Rightarrow (A \Rightarrow A)) \)

Rule \((r)\) application 1 for \( A = A, B = A \)

\( A_3 = (((A \Rightarrow A) \Rightarrow (A \Rightarrow (A \Rightarrow A)))) \)

Rule \((r)\) application 2 for \( A = A, B = (A \Rightarrow A) \)
Soudness

Given a proof system:

\[ S = (\mathcal{L}_{\neg, \Rightarrow}, \mathcal{F}, \{(A \Rightarrow A), (A \Rightarrow (\neg A \Rightarrow B))\}, (r) \frac{(A \Rightarrow B)}{(B \Rightarrow (A \Rightarrow B))}) \]

Problem 2

Prove that \( S \) is sound under classical semantics.

Solution
1. Both axioms of \( S \) are basic classical tautologies
2. Consider the rule of inference of \( S \)

\[
(r) \frac{(A \Rightarrow B)}{(B \Rightarrow (A \Rightarrow B))}
\]

Assume that its premise (the only premise) is true, i.e. let \( v \) be any truth assignment, such that \( v^*(A \Rightarrow B) = T \)

We evaluate logical value of the conclusion under the truth assignment \( v \) as follows

\[
v^*(B \Rightarrow (A \Rightarrow B)) = v^*(B) \Rightarrow T = T
\]

for any \( B \) and any value of \( v^*(B) \)
Formal Proof

Given a proof system:

\[ S = (\mathcal{L}_{\land, \Rightarrow}, \mathcal{F}, \{(A \Rightarrow A), (A \Rightarrow (\neg A \Rightarrow B))\}, (r) \quad \frac{(A \Rightarrow B)}{(B \Rightarrow (A \Rightarrow B))}) \]

Problem 3.
Write a **formal proof** of your choice in \( S \) with 2 applications of the rule \((r)\)

Solution

There many of such proofs, of different length, with different choice if axioms - here is my choice: \( A_1, A_2, A_3 \), where

\( A_1 = (A \Rightarrow A) \)
(Axiom)

\( A_2 = (A \Rightarrow (A \Rightarrow A)) \)

Rule \((r)\) application 1 for \( A = A, B = A \)

\( A_3 = (((A \Rightarrow A) \Rightarrow (A \Rightarrow (A \Rightarrow A)))) \)

Rule \((r)\) application 2 for \( A = A, B = (A \Rightarrow A) \)
Formal Proof

Given a proof system:

\[ S = (\mathcal{L}_{\neg, \Rightarrow}, \mathcal{F}, \{(A \Rightarrow A), (A \Rightarrow (\neg A \Rightarrow B))\}, (r) \begin{align*} \frac{(A \Rightarrow B)}{(B \Rightarrow (A \Rightarrow B))} \end{align*} ) \]

Problem 4

1. Prove, by constructing a formal proof that

\[ \vdash_S ((\neg A \Rightarrow B) \Rightarrow (A \Rightarrow (\neg A \Rightarrow B))) \]

Solution  Required formal proof is a sequence \( A_1, A_2 \), where

\[ A_1 = (A \Rightarrow (\neg A \Rightarrow B)) \]

Axiom

\[ A_2 = (((\neg A \Rightarrow B) \Rightarrow (A \Rightarrow (\neg A \Rightarrow B)))) \]

Rule \((r)\) application for \( A = A, B = (\neg A \Rightarrow B) \)
Soundness Theorem

2. Does above point 1. prove that

$$
\vdash ((\neg A \Rightarrow B) \Rightarrow (A \Rightarrow (\neg A \Rightarrow B)))
$$

Solution

Yes, it does because the system $S$ is sound and we proved by Mathematical Induction over the length of a proof that if $S$ is sound, then the Soundness Theorem holds for $S$. 

Soundness

Problem 5
Given a proof system:

\[ S = ( \mathcal{L}_{\neg, \Rightarrow}, \mathcal{F}, \{(A \Rightarrow A), (A \Rightarrow (\neg A \Rightarrow B))\}, (r) \frac{(A \Rightarrow B)}{B \Rightarrow (A \Rightarrow B)}) \]

Prove that \( S \) is not sound under \( K \) semantics

Solution
Axiom \((A \Rightarrow A)\) is not a \( K \) semantics tautology
Any truth assignment \( v \) such that \( v^*(A) = \bot \) is a counter-model for it
This proves that \( S \) is not sound under \( K \) semantics