cse371/math371
LOGIC

Professor Anita Wasilewska
LECTURE 3e
Chapter 3 REVIEW
Some Definitions and Problems
SOME DEFINITIONS: Part One

There are some basic **Definitions** and sample **Questions** with Solutions from **Chapter 3**

**Study** them them for **MIDTERM**

Knowing all basic **Definitions** is the first step for understanding the material and solve **Problems**

**Solutions** are very carefully written - so you could understand them step by step and hence correctly write yours, which do not need to be that detailed
DEFINITIONS: Propositional Extensional Semantics

Definition 1

Given a propositional language $\mathcal{L}_{	ext{CON}}$ for the set $\text{CON} = C_1 \cup C_2$, where $C_1, C_2$ are respectively the sets of unary and binary connectives.

Let $V$ be a non-empty set of logical values.

Connectives $\triangledown \in C_1$, $\circ \in C_2$ are called extensional iff their semantics is defined by respective functions

\[ \triangledown : V \to V \]  and  \[ \circ : V \times V \to V \]
DEFINITIONS: Propositional Extensional Semantics

Definition 2

Formal definition of a propositional extensional semantics for a given language $\mathcal{L}_{CON}$ consists of providing definitions of the following four main components:

1. Logical Connectives
2. Truth Assignment
3. Satisfaction, Model, Counter-Model
4. Tautology
CLASSICAL PROPOSITIONAL SEMANTICS
DEFINITIONS: Truth Assignment Extension $v^*$

Definition 3
The Language: $\mathcal{L} = \mathcal{L}_{\neg, \Rightarrow, \cup, \cap}$

Given the truth assignment $v : \text{VAR} \rightarrow \{T, F\}$ in classical semantics for the language $\mathcal{L} = \mathcal{L}_{\neg, \Rightarrow, \cup, \cap}$, we define its extension $v^*$ to the set $\mathcal{F}$ of all formulas of $\mathcal{L}$ as $v^* : \mathcal{F} \rightarrow \{T, F\}$ such that

(i) for any $a \in \text{VAR}$

$$v^*(a) = v(a)$$

(ii) and for any $A, B \in \mathcal{F}$ we put

$$v^*(\neg A) = \neg v^*(A)$$

$$v^*((A \cap B)) = \cap(v^*(A), v^*(B))$$

$$v^*((A \cup B)) = \cup(v^*(A), v^*(B))$$

$$v^*((A \Rightarrow B)) = \Rightarrow(v^*(A), v^*(B))$$

$$v^*((A \Leftrightarrow B)) = \Leftrightarrow(v^*(A), v^*(B))$$
DEFINITIONS: Truth Assignment Extension $v^*$ Revisited

Notation
For binary connectives (two argument functions) we adopt a convention to write the symbol of the connective (name of the 2 argument function) between its arguments as we do in a case arithmetic operations.

The condition (ii) of the definition of the extension $v^*$ can be hence written as follows:

(ii) and for any $A, B \in \mathcal{F}$ we put

\[
\begin{align*}
    v^*(\neg A) &= \neg v^*(A); \\
    v^*((A \cap B)) &= v^*(A) \cap v^*(B); \\
    v^*((A \cup B)) &= v^*(A) \cup v^*(B); \\
    v^*((A \Rightarrow B)) &= v^*(A) \Rightarrow v^*(B); \\
    v^*((A \Leftrightarrow B)) &= v^*(A) \Leftrightarrow v^*(B)
\end{align*}
\]
DEFINITIONS: Satisfaction Relation

Definition 4  Let  $v : VAR \rightarrow \{T, F\}$
We say that $v$ satisfies a formula $A \in \mathcal{F}$ iff $v^*(A) = T$

Notation:  $v \models A$
We say that $v$ does not satisfy a formula $A \in \mathcal{F}$ iff $v^*(A) \neq T$

Notation:  $v \not\models A$
DEFINITIONS: Model, Counter-Model, Classical Tautology

Definition 5
Given a formula $A \in \mathcal{F}$ and $v: \text{VAR} \rightarrow \{T, F\}$
We say that
$v$ is a model for $A$ iff $v \models A$
$v$ is a counter-model for $A$ iff $v \not\models A$

Definition 6
$A$ is a tautology iff for any $v: \text{VAR} \rightarrow \{T, F\}$ we have that $v \models A$

Notation
We write symbolically $\models A$ to denote that $A$ is a classical tautology
DEFINITIONS: Restricted Truth Assignments

Notation: for any formula $A$, we denote by $\text{VAR}_A$ a set of all variables that appear in $A$.

Definition 7 Given a formula $A \in \mathcal{F}$, any function

$$v_A : \text{VAR}_A \rightarrow \{T, F\}$$

is called a truth assignment restricted to $A$. 
DEFINITIONS: Restricted Model, Counter Model

Notation: for any formula $A$, we denote by $VAR_A$ a set of all variables that appear in $A$

Definition 8  Given a formula $A \in \mathcal{F}$
Any function

$$w : \ VAR_A \longrightarrow \{T, F\} \text{ such that } w^*(A) = T$$

is called a restricted MODEL for $A$

Any function

$$w : \ VAR_A \longrightarrow \{T, F\} \text{ such that } w^*(A) \neq T$$

is called a restricted Counter- MODEL for $A$
DEFINITIONS: Models for Sets of Formulas

Consider $\mathcal{L} = \mathcal{L}\{\neg, \cup, \cap, \Rightarrow\}$ and let $S \neq \emptyset$ be any non empty set of formulas of $\mathcal{L}$, i.e.

$$S \subseteq \mathcal{F}$$

**Definition 9**

A truth truth assignment $v : VAR \longrightarrow \{T, F\}$ is a **model for the set** $S$ of formulas if and only if

$$v \models A \text{ for all formulas } A \in S$$

We write

$$v \models S$$

to denote that $v$ is a model for the set $S$ of formulas.
DEFINITIONS: Consistent Sets of Formulas

Definition 10
A non-empty set $G \subseteq \mathcal{F}$ of formulas is called **consistent** if and only if $G$ has a model, i.e. we have that

$G \subseteq \mathcal{F}$ is **consistent** if and only if

there is $v$ such that $v \models G$

Otherwise $G$ is called **inconsistent**
DEFINITIONS: Independent Statements

Definition 11
A formula $A$ is called **independent** from a non-empty set $G \subseteq \mathcal{F}$ if and only if there are truth assignments $v_1, v_2$ such that

$$v_1 \models G \cup \{A\} \quad \text{and} \quad v_2 \models G \cup \{\neg A\}$$

i.e. we say that a formula $A$ is **independent** if and only if

$$G \cup \{A\} \quad \text{and} \quad G \cup \{\neg A\} \quad \text{are consistent}$$
Many Valued Extensional Semantics M
DEFINITIONS: Semantics $M$

Definition 11
The extensional semantics $M$ is defined for a non-empty set of $V$ of *logical values of any cardinality*
We only *assume* that the set $V$ of logical values of $M$ always has a special, distinguished logical value which serves to define a notion of tautology
We denote this distinguished value as $T$

Formal definition of *many valued extensional semantics* $M$ for the language $L_{CON}$ consists of giving *definitions* of the following main components:
1. Logical Connectives under semantics $M$
2. Truth Assignment for $M$
3. Satisfaction Relation, Model, Counter-Model under semantics $M$
4. Tautology under semantics $M$
Definition of M - Extensional Connectives

Given a propositional language $\mathcal{L}_{CON}$ for the set $CON = C_1 \cup C_2$, where $C_1$ is the set of all unary connectives, and $C_2$ is the set of all binary connectives.

Let $V$ be a non-empty set of logical values adopted by the semantics $M$.

Definition 12

Connectives $\nabla \in C_1$, $\circ \in C_2$ are called M-extensional iff their semantics $M$ is defined by respective functions

$$\nabla : V \to V \quad \text{and} \quad \circ : V \times V \to V$$
DEFINITION: Definability of Connectives under a semantics $M$

Given a propositional language $L_{CON}$ and its extensional semantics $M$

We adopt the following definition

Definition 13

A connective $\circ \in CON$ is **definable** in terms of some connectives $\circ_1, \circ_2, ... \circ_n \in CON$ for $n \geq 1$ under the semantics $M$ if and only if the connective $\circ$ is a certain function composition of functions $\circ_1, \circ_2, ... \circ_n$ as they are defined by the semantics $M$
Definition 14

Given the \( \textbf{M} \) truth assignment \( v : \text{VAR} \rightarrow V \)

We define its \textbf{M extension} \( v^* \) to the set \( F \) of all formulas of \( \mathcal{L} \) as any function \( v^* : F \rightarrow V \), such that the following conditions are satisfied

(i) for any \( a \in \text{VAR} \)

\[ v^*(a) = v(a); \]

(ii) For any connectives \( \triangledown \in \mathcal{C}_1, \circ \in \mathcal{C}_2 \) and for any formulas \( A, B \in F \) we put

\[ v^*(\triangledown A) = \triangledown v^*(A) \]

\[ v^*((A \circ B)) = \circ(v^*(A), v^*(B)) \]
DEFINITION: M Satisfaction, Model, Counter Model, Tautology

Definition 15  Let \( v : \text{VAR} \rightarrow V \)
Let \( T \in V \) be the distinguished logical value
We say that 
\( v \models M A \) satisfies a formula \( A \in F \) (\( v \models M A \)) iff 
\( v^*(A) = T \)

Definition 16
Given a formula \( A \in F \) and \( v : \text{VAR} \rightarrow V \)
Any \( v \) such that \( v \models M A \) is called a M model for \( A \)
Any \( v \) such that \( v \not\models M A \) is called a M counter model for \( A \)
\( A \) is a M tautology (\( \models M A \)) iff \( v \models M A \), for all \( v : \text{VAR} \rightarrow V \)
CHAPTER 3: Some Sample Questions with Solutions
Chapter 3: Question 1

Question 1
Find a restricted model for formula $A$, where

$$A = (\neg a \Rightarrow (\neg b \cup (b \Rightarrow \neg c)))$$

You can’t use short-hand notation
Show each step of solution

Solution
For any formula $A$, we denote by $\text{VAR}_A$ a set of all variables that appear in $A$
In our case we have $\text{VAR}_A = \{a, b, c\}$
Any function $\nu_A : \text{VAR}_A \rightarrow \{T, F\}$ is called a truth assignment restricted to $A$
Chapter 3: Question 1

Let \( \nu : \text{VAR} \rightarrow \{T, F\} \) be any truth assignment such that

\[
\nu(a) = \nu_A(a) = T, \quad \nu(b) = \nu_A(b) = T, \quad \nu(c) = \nu_A(c) = F
\]

We evaluate the value of the extension \( \nu^* \) of \( \nu \) on the formula \( A \) as follows

\[
\nu^*(A) = \nu^*((\neg a \Rightarrow (\neg b \cup (b \Rightarrow \neg c))))
\]

\[
= \nu^*(\neg a) \Rightarrow \nu^*((\neg b \cup (b \Rightarrow \neg c)))
\]

\[
= \neg \nu^*(a) \Rightarrow (\nu^*(\neg b) \cup \nu^*((b \Rightarrow \neg c)))
\]

\[
= \neg \nu(a) \Rightarrow (\neg \nu(b) \cup (\nu(b) \Rightarrow \neg \nu(c)))
\]

\[
= \neg \nu_A(a) \Rightarrow (\neg \nu_A(b) \cup (\nu_A(b) \Rightarrow \neg \nu_A(c)))
\]

\[
(\neg T \Rightarrow (\neg T \cup (T \Rightarrow \neg F))) = F \Rightarrow (F \cup T) = F \Rightarrow T = T, \text{ i.e.}
\]

\[
\nu_A \models A \quad \text{and} \quad \nu \models A
\]
Chapter 3: Question 2

Question 2
Find a restricted model and a restricted counter-model for $A$, where

$$A = (\neg a \Rightarrow (\neg b \cup (b \Rightarrow \neg c)))$$

You can use short-hand notation. Show work.

Solution
Notation: for any formula $A$, we denote by $\text{VAR}_A$ a set of all variables that appear in $A$. In our case we have $\text{VAR}_A = \{a, b, c\}$.

Any function $v_A : \text{VAR}_A \rightarrow \{T, F\}$ is called a truth assignment restricted to $A$.

We define now $v_A(a) = T$, $v_A(b) = T$, $v_A(c) = F$, in shorthand: $a = T$, $b = T$, $c = F$ and evaluate

$$(\neg T \Rightarrow (\neg T \cup (T \Rightarrow \neg F))) = F \Rightarrow (F \cup T) = F \Rightarrow T = T,$$

i.e.

$$v_A \models A$$
Chapter 3: Question 2

**Observe** that

\((\neg a \Rightarrow (\neg b \cup (b \Rightarrow \neg c))) = T\) when \(a = T\) and \(b, c\) any truth values as by definition of implication we have that \(F \Rightarrow \text{anything} = T\)

Hence \(a = T\) gives us 4 models as we have \(2^2\) possible values on \(b\) and \(c\)
Chapter 3: Question 2

We take as a **restricted counter-model**: \( a = F, \ b = T \) and \( c = T \)

**Evaluation:** observe that
\[
(\neg a \Rightarrow (\neg b \cup (b \Rightarrow \neg c))) = F \quad \text{if and only if}
\]
\[
\neg a = T \quad \text{and} \quad (\neg b \cup (b \Rightarrow \neg c)) = F \quad \text{if and only if}
\]
\[
a = F, \ \neg b = F \quad \text{and} \quad (b \Rightarrow \neg c) = F \quad \text{if and only if}
\]
\[
a = F, b = T \quad \text{and} \quad (T \Rightarrow \neg c) = F \quad \text{if and only if}
\]
\[
a = F, b = T \quad \text{and} \quad \neg c = F \quad \text{if and only if}
\]
\[
a = F, b = T \quad \text{and} \quad c = T
\]
The above proves also that \( a = F, b = T \) and \( c = T \) is the only **restricted counter-model** for \( A \).
Question 3  Justify whether the following statements true or false

S1  There are more then 3 possible restricted counter-models for $A$
S2  There are more then 2 possible restricted models of $A$

Solution

S1 Statement: There are more then 3 possible restricted counter-models for $A$ is false

We have just proved that there is only one possible restricted counter-model for $A$

S2 Statement: There are more then 2 possible restricted models of $A$ is true

There are 7 possible restricted models for $A$

Justification: $2^3 - 1 = 7$
Chapter 3: Question 4

Question 4

1. List 3 models for $A$ from Question 2, i.e. for formula

   \[ A = (\neg a \Rightarrow (\neg b \cup (b \Rightarrow \neg c))) \]

   that are extensions to the set $VAR$ of all variables of one of the restricted models that you have found in Questions 1,

2. List 2 counter models for $A$ that are extensions of one of the restricted counter models that you have found in the Questions 1, 2
Chapter 3: Question 4

Solution

1. One of the **restricted models** is, for example a function $\nu_A : \{a, b, c\} \rightarrow \{T, F\}$ such that $\nu_A(a) = T$, $\nu_A(b) = T$, $\nu_A(c) = F$

We extend $\nu_A$ to the set of all propositional variables $VAR$ to obtain a (non restricted) **models** as follows
Chapter 3: Question 4

Model \( w_1 \) is a function

\[
w_1 : \text{VAR} \rightarrow \{ T, F \}
\]
such that

\[
w_1(a) = v_A(a) = T, \quad w_1(b) = v_A(b) = T, \\
w_1(c) = v_A(c) = F, \quad \text{and} \quad w_1(x) = T, \quad \text{for all} \quad x \in \text{VAR} - \{a, b, c\}
\]

Model \( w_2 \) is defined by a formula

\[
w_2(a) = v_A(a) = T, \quad w_2(b) = v_A(b) = T, \\
w_2(c) = v_A(c) = F, \quad \text{and} \quad w_2(x) = F, \quad \text{for all} \quad x \in \text{VAR} - \{a, b, c\}
\]
Chapter 3: Question 4

Model $w_3$ is defined by a formula $w_3(a) = v_A(a) = T$, $w_3(b) = v_A(b) = T$, $w_3(c) = v(c) = F$, $w_3(d) = F$ and $w_3(x) = T$ for all $x \in VAR - \{a, b, c, d\}$

There is as many of such models, as extensions of $v_A$ to the set $VAR$, i.e. as many as real numbers
Chapter 3: Question 4

2. A **counter-model** for a formula
   \[ A = (\neg a \Rightarrow (\neg b \cup (b \Rightarrow \neg c)) \] is, by **definition** any function
   \[ v : \text{VAR} \rightarrow \{T, F\} \]
   such that \[ v^*(A) = F \]
   A **restricted counter-model** for the formula \( A \), the only one, as already proved in is a function
   \[ v_A : \{a, b\} \rightarrow \{T, F\} \]
   such that
   \[ v_A(a) = F, \quad v_A(b) = T, \quad v_A(c) = T \]
Chapter 3: Question 4

We extend $v_A$ to the set of all propositional variables $VAR$ to obtain (non restricted) some counter-models. Here are two of such extensions.

**Counter-model $w_1$:**

- $w_1(a) = v_A(a) = F$
- $w_1(b) = v_A(b) = T$
- $w_1(c) = v(c) = T$
- $w_1(x) = F$ for all $x \in VAR - \{a, b, c\}$

**Counter-model $w_2$:**

- $w_2(a) = v_A(a) = T$
- $w_2(b) = v_A(b) = T$
- $w_2(c) = v(c) = T$
- $w_2(x) = T$ for all $x \in VAR - \{a, b, c\}$

There is as many of such counter-models, as extensions of $v_A$ to the set $VAR$, i.e. as many as real numbers.
Chapter 3: Models for Sets of Formulas

Definition
A truth assignment $v$ is a **model for a set** $G \subseteq \mathcal{F}$ of formulas of a given language $\mathcal{L} = \mathcal{L}\{\neg, \Rightarrow, \cup, \cap\}$ if and only if
\[
v \models B \quad \text{for all} \quad B \in G
\]
We denote it by $v \models G$.

Observe that the set $G \subseteq \mathcal{F}$ can be **finite** or **infinite**.
Chapter 3: Consistent Sets of Formulas

Definition
A set \( G \subseteq F \) of formulas is called **consistent** if and only if \( G \) has a model, i.e. we have that

\[
G \subseteq F \text{ is consistent if and only if there is } v \text{ such that } v \models G
\]

Otherwise \( G \) is called **inconsistent**
Chapter 3: Independent Statements

Definition
A formula $A$ is called **independent** from a set $G \subseteq \mathcal{F}$ if and only if there are truth assignments $v_1, v_2$ such that

$$v_1 \models G \cup \{A\} \quad \text{and} \quad v_2 \models G \cup \{\neg A\}$$

i.e. we say that a formula $A$ is **independent** if and only if

$$G \cup \{A\} \quad \text{and} \quad G \cup \{\neg A\} \quad \text{are consistent}$$
Chapter 3: Question 5

Question 5
Given a set \( G = \{((a \cap b) \Rightarrow b), (a \cup b), \neg a\} \)
Show that \( G \) is consistent

Solution
We have to find \( v : \text{VAR} \longrightarrow \{T, F\} \) such that \( v \models G \)
It means that we need to find a \( v \) such that

\[
v^*((a \cap b) \Rightarrow b) = T, \quad v^*(a \cup b) = T, \quad v^*(\neg a) = T
\]

We write it in the shorthand notation

\[
((a \cap b) \Rightarrow b) = T, \quad (a \cup b) = T, \quad \neg a = T
\]

We have to find out if it is possible
Chapter 3: Question 5

1. Observe that \( \models ((a \cap b) \Rightarrow b) \), hence we have that 
\( v^*((a \cap b) \Rightarrow b) = T \) for any \( v \)

2. Case \( \neg a = T \) holds if and only if \( a = F \)

3. Case \( (a \cup b) = T \) holds if and only if \( (T \cup b) = T \) as \( a = F \), and this holds if and only if \( b = T \)

This proves that for any \( v : \text{VAR} \rightarrow \{T, F\} \) such that \( v(a) = F, \ v(b) = T \), is a model for \( G \) and so, by definition, that \( G \) is consistent

Moreover, we have proved that it is the only (restricted) model for \( G \)
Chapter 3: Question 6

Question 6
Show that a formula  \( A = (\neg a \cap b) \) is not independent of

\[ G = \{ ((a \cap b) \Rightarrow b), (a \cup b), \neg a \} \]

Solution
We have to show that it is impossible to construct \( v_1, v_2 \) such that

\[ v_1 \models G \cup \{A\} \quad \text{and} \quad v_2 \models G \cup \{\neg A\} \]

Observe that we have just proved that any \( v \) such that \( v(a) = F, \) and \( v(b) = T \) is the only model restricted to the set of variables \( \{a, b\} \) for \( G \) so we have to check now if it is possible that for that formula \( A = (\neg a \cap b), \) \( v \models A \) and \( v \models \neg A \)
Chapter 3: Question 6

We have to evaluate \( v^*(A) \) and \( v^*(\neg A) \) for

\[ v(a) = F, \quad \text{and} \quad v(b) = T \]

\[ v^*(A) = v^*((\neg a \cap b)) = \neg v(a) \cap v(b) = \neg F \cap T = T \cap T = T \]

and so \( v \models A \)

\[ v^*(\neg A) = \neg v^*(A) = \neg T = F \]

and so \( v \not\models \neg A \)

This ends the proof that \( A \) is not independent of \( G \)
Chapter 3: Question 7

Question 7
Find an infinite number of formulas that are independent of

\[ G = \{ ((a \land b) \Rightarrow b), (a \lor b), \neg a \} \]

This my solution - there are many others, but this one seemed to me to be the simplest

Solution
We just proved that any \( v \) such that \( v(a) = F, \ v(b) = T \) is the only model restricted to the set of variables \( \{a, b\} \) and so all other possible models for \( G \) must be extensions of \( v \)
We define a countably infinite set of formulas (and their negations) and corresponding extensions of \( v \) (restricted to to the set of variables \( \{a, b\} \)) such that \( v \models G \) as follows.

Observe that all extensions of \( v \) restricted to to the set of variables \( \{a, b\} \) have as domain the infinitely countable set

\[
\text{VAR} - \{a, b\} = \{a_1, a_2, \ldots, a_n, \ldots\}
\]

We take as a set of formulas (to be proved to be independent) the set of atomic formulas

\[
\mathcal{F}_0 = \text{VAR} - \{a, b\} = \{a_1, a_2, \ldots, a_n, \ldots\}
\]
proof of independence of any formula of $\mathcal{F}_0$

Let $c \in \mathcal{F}_0$

We define truth assignments $v_1, v_2 : \text{VAR} \rightarrow \{T, F\}$ such that

$v_1 \models G \cup \{c\}$ and $v_2 \models G \cup \{\neg c\}$

as follows

$v_1(a) = v(a) = F, \quad v_1(b) = v(b) = T$ and $v_1(c) = T$

for all $c \in \mathcal{F}_0$

$v_2(a) = v(a) = F, \quad v_2(b) = v(b) = T$ and $v_2(c) = F$

for all $c \in \mathcal{F}_0$
CHAPTER 3
Some Extensional Many Valued Semantics
Chapter 3: Question 8

Question 8
We define a 4 valued $H_4$ logic semantics as follows

The language is $\mathcal{L} = \mathcal{L}\{\neg, \Rightarrow, \cup, \cap\}$

The logical connectives $\neg$, $\Rightarrow$, $\cup$, $\cap$ of $H_4$ are operations in the set $\{F, \bot_1, \bot_2, T\}$, where $\{F < \bot_1 < \bot_2 < T\}$ and are defined as follows

**Conjunction** $\cap$ is a function

$\cap : \{F, \bot_1, \bot_2, T\} \times \{F, \bot_1, \bot_2, T\} \rightarrow \{F, \bot_1, \bot_2, , T\}$, such that for any $x, y \in \{F, \bot_1, \bot_2, T\}$

$$x \cap y = \min\{x, y\}$$
Chapter 3: Question 8

Disjunction  $\cup$  is a function
$\cup : \{F, \bot_1, \bot_2, T\} \times \{F, \bot_1, \bot_2, T\} \rightarrow \{F, \bot_1, \bot_2, T\}$, such that for any $x, y \in \{F, \bot_1, \bot_2, T\}$

$$x \cup y = \max\{x, y\}$$

Implication  $\Rightarrow$  is a function
$\Rightarrow : \{F, \bot_1, \bot_2, T\} \times \{F, \bot_1, \bot_2, T\} \rightarrow \{F, \bot_1, \bot_2, T\}$, such that for any $x, y \in \{F, \bot_1, \bot_2, T\}$,

$$x \Rightarrow y = \begin{cases} T & \text{if } x \leq y \\ y & \text{otherwise} \end{cases}$$

Negation: for any $x, y \in \{F, \bot_1, \bot_2, T\}$

$$\neg x = x \Rightarrow F$$
Chapter 3: Question 8

Part 1  Write Truth Tables for IMPLICATION and NEGATION in $H_4$

Solution

$H_4$ Implication

<table>
<thead>
<tr>
<th>$\Rightarrow$</th>
<th>F</th>
<th>$\bot_1$</th>
<th>$\bot_2$</th>
<th>T</th>
</tr>
</thead>
<tbody>
<tr>
<td>F</td>
<td>T</td>
<td>T</td>
<td>T</td>
<td>T</td>
</tr>
<tr>
<td>$\bot_1$</td>
<td>F</td>
<td>T</td>
<td>T</td>
<td>T</td>
</tr>
<tr>
<td>$\bot_2$</td>
<td>F</td>
<td>$\bot_1$</td>
<td>T</td>
<td>T</td>
</tr>
<tr>
<td>T</td>
<td>F</td>
<td>$\bot_1$</td>
<td>$\bot_2$</td>
<td>T</td>
</tr>
</tbody>
</table>

$H_4$ Negation

<table>
<thead>
<tr>
<th>$\neg$</th>
<th>F</th>
<th>$\bot_1$</th>
<th>$\bot_2$</th>
<th>T</th>
</tr>
</thead>
<tbody>
<tr>
<td>T</td>
<td>F</td>
<td>T</td>
<td>F</td>
<td>F</td>
</tr>
</tbody>
</table>
Chapter 3: Question 7

Part 2   Verify whether

$$\models_{H_4} ((a \Rightarrow b) \Rightarrow (\neg a \cup b))$$

Solution
Take any \(v\) such that  
\(v(a) = \bot_1\quad v(b) = \bot_2\)

Evaluate  
\(v \ast ((a \Rightarrow b) \Rightarrow (\neg a \cup b)) = (\bot_1 \Rightarrow \bot_2) \Rightarrow (\neg \bot_1 \cup \bot_2) = T \Rightarrow (F \cup \bot_2)) = T \Rightarrow \bot_2 = \bot_2\)

This proves that our \(v\) is a counter-model and hence

$$\not\models_{H_4} ((a \Rightarrow b) \Rightarrow (\neg a \cup b))$$
Chapter 3: Question 9

Question 9
Show that (can’t use TTables!)

\[ \vdash ((\neg a \cup b) \Rightarrow (((c \cap d) \Rightarrow \neg d) \Rightarrow (\neg a \cup b))) \]

Solution
Denote \( A = (\neg a \cup b) \), and \( B = ((c \cap d) \Rightarrow \neg d) \)

Our formula becomes a substitution of a **basic tautology**

\[ (A \Rightarrow (B \Rightarrow A)) \]

and hence is a **tautology**
Chapter 3: Challenge Exercise

1. Define your own propositional language $L_{CON}$ that contains also different connectives that the standard connectives $\neg$, $\cup$, $\cap$, $\Rightarrow$

Your language $L_{CON}$ does not need to include all (if any!) of the standard connectives $\neg$, $\cup$, $\cap$, $\Rightarrow$

2. Describe intuitive meaning of the new connectives of your language

3. Give some motivation for your own semantic

4. Define formally your own extensional semantics $M$ for your language $L_{CON}$ - it means write carefully all Steps 1-4 of the definition of your $M$
Chapter 3: Question 10

Question 10

Definition

Let $S_3$ be a 3-valued semantics for $L_{\neg, \lor, \Rightarrow}$ defined as follows:

$V = \{F, U, T\}$ is the set of logical values with the distinguished value $T$

$x \Rightarrow y = \neg x \lor y$ for any $x, y \in \{F, U, T\}$

$\neg F = T$, $\neg U = F$, $\neg T = U$

and

<table>
<thead>
<tr>
<th>$\lor$</th>
<th>$F$</th>
<th>$U$</th>
<th>$T$</th>
</tr>
</thead>
<tbody>
<tr>
<td>$F$</td>
<td>$F$</td>
<td>$U$</td>
<td>$T$</td>
</tr>
<tr>
<td>$U$</td>
<td>$U$</td>
<td>$U$</td>
<td>$U$</td>
</tr>
<tr>
<td>$T$</td>
<td>$T$</td>
<td>$U$</td>
<td>$T$</td>
</tr>
</tbody>
</table>
Part 1
Consider the following classical tautologies:

\[ A_1 = (a \cup \neg a), \quad A_2 = (a \Rightarrow (b \Rightarrow a)) \]

Find \( S_3 \) counter-models for \( A_1, A_2 \), if exist

You can’t use shorthand notation

Solution
Any \( v \) such that \( v(a) = v(b) = U \) is a counter-model for both \( A_1 \) and \( A_2 \), as

\[
\begin{align*}
 v^*(a \cup \neg a) &= v^*(a) \cup \neg v^*(b) = U \cup \neg U = U \cup F = U \neq T \\
 v^*(a \Rightarrow (b \Rightarrow a)) &= v^*(a) \Rightarrow (v^*(b) \Rightarrow v^*(a)) = U \Rightarrow (U \Rightarrow U) = U \Rightarrow U = \neg U \cup U = F \cup U = U \neq T
\end{align*}
\]
Question 10

Part 2
Consider the following classical tautologies:

\[ A_1 = (a \cup \neg a), \quad A_2 = (a \Rightarrow (b \Rightarrow a)) \]

Define your own 2-valued semantics \( S_2 \) for \( L \), such that none of \( A_1, A_2 \) is a \( S_2 \) tautology

Verify your results. You can use shorthand notation.

Solution
This is not the only solution, but it is the simplest and most obvious I could think of! Here it is.

We define \( S_2 \) connectives as follows:

\[ \neg x = F, \quad x \Rightarrow y = F, \quad x \cup y = F \] for all \( x, y \in \{F, T\} \)

Obviously, for any \( v \),

\[ v^*(a \cup \neg a) = F \] and \[ v^*(a \Rightarrow (b \Rightarrow a)) = F \]
Chapter 3: Question 11

Question 11
Prove using proper classical logical equivalences (list them at each step) that for any formulas $A, B$ of language $\mathcal{L}_{\{\neg, \lor, \Rightarrow\}}$

$$\neg(A \Leftrightarrow B) \equiv ((A \cap \neg B) \cup (\neg A \cap B))$$

Solution

$$\neg(A \Leftrightarrow B) \equiv^{def} \neg((A \Rightarrow B) \cap (B \Rightarrow A))$$

$$\equiv^{deMorgan} (\neg(A \Rightarrow B) \cup \neg(B \Rightarrow A))$$

$$\equiv^{negimpl} ((A \cap \neg B) \cup (B \cap \neg A)) \equiv^{commut} ((A \cap \neg B) \cup (\neg A \cap B))$$
Question 12

Prove using proper **classical** logical equivalences (list them at each step) that for any formulas $A, B$ of language $\mathcal{L}_{\{\neg, \cup, \Rightarrow\}}$

$$(((B \cap \neg C) \Rightarrow (\neg A \cup B)) \equiv ((B \Rightarrow C) \cup (A \Rightarrow B))$$

**Solution**

$$(((B \cap \neg C) \Rightarrow (\neg A \cup B))$$

$\equiv impl((\neg (B \cap \neg C) \cup (\neg A \cup B))$

$\equiv deMorgan(((\neg B \cup \neg C) \cup (\neg A \cup B))$

$\equiv dneg(((\neg B \cup C) \cup (\neg A \cup B)) \equiv impl((B \Rightarrow C) \cup (A \Rightarrow B))$$
We define Ł connectives for \( \mathcal{L}\{\neg, \cup, \Rightarrow\} \) as follows

Ł **Negation** \( \neg \) is a function:

\[
\neg : \{T, \bot, F\} \rightarrow \{T, \bot, F\}
\]

such that \( \neg \bot = \bot, \ \neg T = F, \ \neg F = T \)

Ł **Conjunction** \( \cap \) is a function:

\[
\cap : \{T, \bot, F\} \times \{T, \bot, F\} \rightarrow \{T, \bot, F\}
\]

such that \( x \cap y = \min\{x, y\} \) for all \( x, y \in \{T, \bot, F\} \)

Remember that we assumed: \( F < \bot < T \)
Ł Implication ⇒ is a function:

⇒: \{T, \bot, F\} \times \{T, \bot, F\} \rightarrow \{T, \bot, F\}

such that

\[ x \Rightarrow y = \begin{cases} 
\neg x \cup y & \text{if } x > y \\
T & \text{otherwise}
\end{cases} \]

Given a formula \(((a \cap b) \Rightarrow \neg b) \in \mathcal{F}\) of \(\mathcal{L}_{\neg, \cup, \Rightarrow}\)

**Use the fact** that \(v : \text{VAR} \rightarrow \{F, \bot, T\}\) is such that

\[ v^*((((a \cap b) \Rightarrow \neg b)) = \bot \] under Ł semantics to evaluate all possible \(v^*((((b \Rightarrow \neg a) \Rightarrow (a \Rightarrow \neg b))) \cup (a \Rightarrow b))\)

You can use shorthand notation
Question 13 Solution

Solution

The formula $((a \cap b) \Rightarrow \neg b) = \bot$ in Ł connectives semantics in two cases written is the shorthand notation as

**C1** \((a \cap b) = \bot\) and \(\neg b = F\)

**C2** \((a \cap b) = T\) and \(\neg b = \bot\).

Consider case **C1**

\(\neg b = F\), so \(v(b) = T\), and hence \((a \cap T) = v(a) \cap T = \bot\)

if and only if \(v(a) = \bot\)

It means that \(v^*((((a \cap b) \Rightarrow \neg b))) = \bot\) for any \(v\), is such that \(v(a) = \bot\) and \(v(b) = T\)
We now evaluate (in shorthand notation)

\[ v^*(((b \Rightarrow \neg a) \Rightarrow (a \Rightarrow \neg b)) \cup (a \Rightarrow b)) \]

\[ = (((T \Rightarrow \neg \bot) \Rightarrow (\bot \Rightarrow \neg T)) \cup (\bot \Rightarrow T)) = ((\bot \Rightarrow \bot) \cup T) = T \]

Consider now Case C2

\[ \neg b = \bot, \text{ i.e. } b = \bot, \text{ and hence } (a \cap \bot) = T \text{ what is impossible, hence } v \text{ from the Case C1 is the only one} \]
Use the **Definability of Conjunction** in terms of disjunction and negation **Equivalence**

\[(A \cap B) \equiv \neg(\neg A \cup \neg B)\]

to transform a formula

\[A = \neg(\neg(\neg a \cap \neg b) \cap a)\]

of the language \(\mathcal{L}_{\{\cap, \neg}\}\) into a logically equivalent formula \(B\) of the language \(\mathcal{L}_{\{\cup, \neg}\}\)
Question 14

Solution

\[ \neg(\neg(a \cap \neg b) \cap a) \equiv \neg(\neg(a \cap \neg b) \cup \neg a) \]

\[ \equiv ((\neg a \cap \neg b) \cup \neg a) \equiv (\neg(a \cup \neg b) \cup \neg a) \]

\[ \equiv \neg(a \cup b) \cup \neg a \]

The formula \( B \) of \( \mathcal{L}_{\{\cup, \neg\}} \) equivalent to \( A \) is

\[ B = (\neg(a \cup b) \cup \neg a) \]
Equivalence of Languages Definition

Definition

Given two languages: \( L_1 = L_{\text{CON}_1} \) and \( L_2 = L_{\text{CON}_2} \), for \( \text{CON}_1 \neq \text{CON}_2 \)

We say that they are logically equivalent, i.e.

\[ L_1 \equiv L_2 \]

if and only if the following conditions \( \textbf{C1}, \textbf{C2} \) hold.

\( \textbf{C1} \): for any formula \( A \) of \( L_1 \), there is a formula \( B \) of \( L_2 \), such that \( A \equiv B \)

\( \textbf{C2} \): for any formula \( C \) of \( L_2 \), there is a formula \( D \) of \( L_1 \), such that \( C \equiv D \)
Question 14

Prove the logical equivalence of the languages

\[ L_{\{\neg, \cup\}} \equiv L_{\{\neg, \Rightarrow\}} \]

Solution

We need **two definability equivalences**: implication in terms of disjunction and negation

\[(A \Rightarrow B) \equiv (\neg A \cup B)\]

and disjunction in terms of implication and negation,

\[(A \cup B) \equiv (\neg A \Rightarrow B)\]

and the **Substitution Theorem**
Question 15

Prove the logical equivalence of the languages

\[ \mathcal{L}_{\{\neg, \cap, \cup, \Rightarrow\}} \equiv \mathcal{L}_{\{\neg, \cap, \cup\}} \]

Solution

We need only the **definability of implication** in terms of disjunction and negation equivalence

\[ (A \Rightarrow B) \equiv (\neg A \cup B) \]

as the **Substitution Theorem** for any formula \( A \) of \( \mathcal{L}_{\{\neg, \cap, \cup, \Rightarrow\}} \)

**there is** a formula \( B \) of \( \mathcal{L}_{\{\neg, \cap, \cup\}} \) such that \( A \equiv B \) and the condition **C1** holds

Observe that any formula \( A \) of language \( \mathcal{L}_{\{\neg, \cap, \cup\}} \) is also a formula of the language \( \mathcal{L}_{\{\neg, \cap, \cup, \Rightarrow\}} \) and of course \( A \equiv A \) so the condition **C2** also holds
Question 16

Prove that

\[ \mathcal{L}_{\neg, \cap} \equiv \mathcal{L}_{\neg, \Rightarrow} \]

Solution

The equivalence of languages holds due to the following two definability of connectives equivalences, respectively

\[ (A \cap B) \equiv \neg(A \Rightarrow \neg B), \quad (A \Rightarrow B) \equiv \neg(A \cap \neg B) \]

and Substitution Theorem
Question 17

Prove that in classical semantics

\[ \mathcal{L}_{\{\neg, \Rightarrow\}} \equiv \mathcal{L}_{\{\neg, \Rightarrow, \cup\}} \]

Solution

OBSERVE that the condition \textbf{C1} holds because any formula of \( \mathcal{L}_{\{\neg, \Rightarrow\}} \) is also a formula of \( \mathcal{L}_{\{\neg, \Rightarrow, \cup\}} \).

Condition \textbf{C2} holds due to the following definability of connectives equivalence

\[(A \cup B) \equiv (\neg A \Rightarrow B)\]

and \textbf{Substitution Theorem}
Question 18

Prove that the equivalence defining $\cup$ in terms of negation and implication in classical logic does not hold under $\mathcal{L}$ semantics, i.e. that

$$(A \cup B) \neq_{\mathcal{L}} (\neg A \Rightarrow B)$$

but nevertheless

$$\mathcal{L}_{\{\neg, \Rightarrow\}} \equiv_{\mathcal{L}} \mathcal{L}_{\{\neg, \Rightarrow, \cup\}}$$
Question 18

Solution

We prove

$$\mathcal{L}_{\neg,\Rightarrow} \equiv_L \mathcal{L}_{\neg,\Rightarrow,\cup}$$

as follows

Condition **C2** holds because the definability of connectives equivalence

$$(A \cup B) \equiv_L ((A \Rightarrow B) \Rightarrow B)$$

Check it by verification as an exercise

**C1** holds because any formula of $\mathcal{L}_{\neg,\Rightarrow}$ is a formula of $\mathcal{L}_{\neg,\Rightarrow,\cup}$

**Observe** that the equivalence $(A \cup B) \equiv (A \Rightarrow B) \Rightarrow B)$ provides also an alternative proof of **C2** in classical case