cse371/math371
LOGIC

Professor Anita Wasilewska
LECTURE 3e
Chapter 3 REVIEW
Some Definitions and Problems
SOME DEFINITIONS: Part One

There are some basic **DEFINITIONS** from Chapter 3

You have to KNOW them for Q1 and MIDTERM

I could ask you to **WRITE** down a full, correct text of 1-3 of them - in EXACTLY the same form as they are presented here, or their particular case

Knowing all basic **Definitions** is the first step for understanding the material
DEFINITIONS: Propositional Extensional Semantics

Definition 1
Given a propositional language $L_{CON}$ for the set $CON = C_1 \cup C_2$, where $C_1, C_2$ are respectively the sets of unary and binary connectives

Let $V$ be a non-empty set of logical values

Connectives $\lor \in C_1$, $\circ \in C_2$ are called extensional iff their semantics is defined by respective functions

$$\lor : V \rightarrow V \quad \text{and} \quad \circ : V \times V \rightarrow V$$
Definition 2

Formal definition of a **propositional extensional semantics** for a given language $\mathcal{L}_{CON}$ consists of providing **definitions** of the following four main components:

1. Logical Connectives
2. Truth Assignment
3. Satisfaction, Model, Counter-Model
4. Tautology
CLASSICAL PROPOSITIONAL SEMANTICS
DEFINITIONS: Truth Assignment Extension $v^*$

Definition 3
The Language: \( \mathcal{L} = \mathcal{L}_{\{\neg, \Rightarrow, \cup, \cap\}} \)

Given the truth assignment \( v : \text{VAR} \rightarrow \{T, F\} \) in classical semantics for the language \( \mathcal{L} = \mathcal{L}_{\{\neg, \Rightarrow, \cup, \cap\}} \)

We define its extension \( v^* \) to the set \( \mathcal{F} \) of all formulas of \( \mathcal{L} \) as \( v^* : \mathcal{F} \rightarrow \{T, F\} \) such that

(i) for any \( a \in \text{VAR} \)

\[
    v^*(a) = v(a)
\]

(ii) and for any \( A, B \in \mathcal{F} \) we put

\[
    v^*(\neg A) = \neg v^*(A);
\]

\[
    v^*((A \cap B)) = \cap(v^*(A), v^*(B));
\]

\[
    v^*((A \cup B)) = \cup(v^*(A), v^*(B));
\]

\[
    v^*((A \Rightarrow B)) = \Rightarrow(v^*(A), v^*(B));
\]

\[
    v^*((A \Leftrightarrow B)) = \Leftrightarrow(v^*(A), v^*(B))
\]
DEFINITIONS: Truth Assignment Extension \( \nu^* \) Revisited

Notation
For binary connectives (two argument functions) we adopt a convention to write the symbol of the connective (name of the 2 argument function) between its arguments as we do in a case arithmetic operations.

The condition (ii) of the definition of the extension \( \nu^* \) can be hence written as follows:

(ii) and for any \( A, B \in F \) we put

\[
\begin{align*}
\nu^*(\neg A) &= \neg \nu^*(A); \\
\nu^*((A \cap B)) &= \nu^*(A) \cap \nu^*(B); \\
\nu^*((A \cup B)) &= \nu^*(A) \cup \nu^*(B); \\
\nu^*((A \Rightarrow B)) &= \nu^*(A) \Rightarrow \nu^*(B); \\
\nu^*((A \Leftrightarrow B)) &= \nu^*(A) \Leftrightarrow \nu^*(B)
\end{align*}
\]
DEFINITIONS: Satisfaction Relation

Definition 4  Let $v : \text{VAR} \rightarrow \{T, F\}$

We say that $v$ satisfies a formula $A \in \mathcal{F}$ iff $v^*(A) = T$

Notation: $v \models A$

We say that $v$ does not satisfy a formula $A \in \mathcal{F}$ iff $v^*(A) \neq T$

Notation: $v \not\models A$
DEFINITIONS: Model, Counter-Model, Classical Tautology

Definition 5
Given a formula $A \in \mathcal{F}$ and $v : \text{VAR} \rightarrow \{T, F\}$
We say that $v$ is a **model** for $A$ iff $v \models A$
$v$ is a **counter-model** for $A$ iff $v \not\models A$

Definition 6
$A$ is a **tautology** iff for any $v : \text{VAR} \rightarrow \{T, F\}$ we have that $v \models A$

Notation
We write symbolically $\models A$ to denote that $A$ is a **classical tautology**
DEFINITIONS: Restricted Truth Assignments

Notation: for any formula $A$, we denote by $VAR_A$ a set of all variables that appear in $A$

Definition 7  Given a formula $A \in \mathcal{F}$, any function

$$v_A : VAR_A \rightarrow \{T, F\}$$

is called a truth assignment restricted to $A$
DEFINITIONS: Restricted Model, Counter Model

Notation: for any formula $A$, we denote by $\text{VAR}_A$ a set of all variables that appear in $A$

Definition 8  Given a formula $A \in \mathcal{F}$

Any function

$$w : \text{VAR}_A \rightarrow \{T, F\} \text{ such that } w^*(A) = T$$

is called a **restricted MODEL** for $A$

Any function

$$w : \text{VAR}_A \rightarrow \{T, F\} \text{ such that } w^*(A) \neq T$$

is called a **restricted Counter- MODEL** for $A$
DEFINITIONS: Models for Sets of Formulas

Consider $\mathcal{L} = \mathcal{L}\{\neg, \cup, \cap, \Rightarrow\}$ and let $S \neq \emptyset$ be any non-empty set of formulas of $\mathcal{L}$, i.e.

$$S \subseteq \mathcal{F}$$

Definition 9
A truth assignment $v : \text{VAR} \rightarrow \{T, F\}$ is a model for the set $S$ of formulas if and only if

$$v \models A \text{ for all formulas } A \in S$$

We write

$$v \models S$$

to denote that $v$ is a model for the set $S$ of formulas.
**DEFINITIONS: Consistent Sets of Formulas**

**Definition 10**

A non-empty set $G \subseteq \mathcal{F}$ of formulas is called **consistent** if and only if $G$ has a model, i.e. we have that

\[ G \subseteq \mathcal{F} \text{ is consistent} \quad \text{if and only if} \quad \text{there is } v \text{ such that } v \models G \]

Otherwise $G$ is called **inconsistent**.
DEFINITIONS: Independent Statements

Definition 11
A formula $A$ is called **independent** from a non-empty set $G \subseteq \mathcal{F}$ if and only if there are truth assignments $v_1, v_2$ such that

$$v_1 \models G \cup \{A\} \quad \text{and} \quad v_2 \models G \cup \{\neg A\}$$

i.e. we say that a formula $A$ is **independent** if and only if

$$G \cup \{A\} \quad \text{and} \quad G \cup \{\neg A\} \quad \text{are consistent}$$
Many Valued Extensional Semantics M
DEFINITIONS: Semantics $M$

Definition 11
The extensional semantics $M$ is defined for a non-empty set of $V$ of logical values of any cardinality.
We only assume that the set $V$ of logical values of $M$ always has a special, distinguished logical value which serves to define a notion of tautology.

We denote this distinguished value as $T$.

Formal definition of many valued extensional semantics $M$ for the language $L_{CON}$ consists of giving definitions of the following main components:

1. Logical Connectives under semantics $M$
2. Truth Assignment for $M$
3. Satisfaction Relation, Model, Counter-Model under semantics $M$
4. Tautology under semantics $M$
Definition of M - Extensional Connectives

Given a propositional language $L_{\text{CON}}$ for the set $\text{CON} = C_1 \cup C_2$, where $C_1$ is the set of all unary connectives, and $C_2$ is the set of all binary connectives

Let $V$ be a non-empty set of logical values adopted by the semantics $M$

**Definition 12**

Connectives $\nabla \in C_1$, $\circ \in C_2$ are called M -extensional iff their semantics $M$ is defined by respective functions

$$\nabla : V \rightarrow V \quad \text{and} \quad \circ : V \times V \rightarrow V$$
DEFINITION: Definability of Connectives under a semantics \( M \)

Given a propositional language \( L_{\text{CON}} \) and its extensional semantics \( M \)

We adopt the following definition

**Definition 13**

A connective \( \circ \in CON \) is **definable** in terms of some connectives \( \circ_1, \circ_2, \ldots, \circ_n \in CON \) for \( n \geq 1 \) under the semantics \( M \) if and only if the connective \( \circ \) is a certain function composition of functions \( \circ_1, \circ_2, \ldots, \circ_n \) as they are defined by the semantics \( M \)
DEFINITION: M Truth Assignment Extension $v^*$ to $\mathcal{F}$

Definition 14

Given the M truth assignment $v : \text{VAR} \rightarrow V$

We define its M extension $v^*$ to the set $\mathcal{F}$ of all formulas of $\mathcal{L}$ as any function $v^* : \mathcal{F} \rightarrow V$, such that the following conditions are satisfied

(i) for any $a \in \text{VAR}$

$$v^*(a) = v(a);$$

(ii) For any connectives $\nabla \in C_1$, $\circ \in C_2$ and for any formulas $A, B \in \mathcal{F}$ we put

$$v^*(\nabla A) = \nabla v^*(A)$$

$$v^*((A \circ B)) = \circ(v^*(A), v^*(B))$$
**Definition 15** Let \( v : \text{VAR} \rightarrow V \)

Let \( T \in V \) be the distinguished logical value

We say that \( v \) \text{ M satisfies} a formula \( A \in \mathcal{F} \) \( (v \models_{M} A) \) iff \( v^*(A) = T \)

**Definition 16**

Given a formula \( A \in \mathcal{F} \) and \( v : \text{VAR} \rightarrow V \)

Any \( v \) such that \( v \models_{M} A \) is called a \text{M model} for \( A \)

Any \( v \) such that \( v \not\models_{M} A \) is called a \text{M counter model} for \( A \)

\( A \) is a \text{M tautology} \( (\models_{M} A) \) iff \( v \models_{M} A \), for all \( v : \text{VAR} \rightarrow V \)
CHAPTER 3: Some Questions
Chapter 3: Question 1

Question 1
1. Find a restricted model for formula $A$, where

$$A = (\neg a \Rightarrow (\neg b \cup (b \Rightarrow \neg c)))$$

You can’t use short-hand notation
Show each step of solution

Solution
For any formula $A$, we denote by $\text{VAR}_A$ a set of all variables that appear in $A$
In our case we have $\text{VAR}_A = \{a, b, c\}$
Any function $v_A: \text{VAR}_A \rightarrow \{T, F\}$ is called a truth assignment restricted to $A$
Chapter 3: Question 1

Let \( v : \text{VAR} \rightarrow \{ T, F \} \) be any truth assignment such that

\[
\begin{align*}
    v(a) &= v_A(a) = T, \\
    v(b) &= v_A(b) = T, \\
    v(c) &= v_A(c) = F
\end{align*}
\]

We evaluate the value of the extension \( v^* \) of \( v \) on the formula \( A \) as follows

\[
\begin{align*}
    v^*(A) &= v^*((\neg a \Rightarrow (\neg b \cup (b \Rightarrow \neg c)))) \\
    &= v^*(\neg a) \Rightarrow v^*((\neg b \cup (b \Rightarrow \neg c))) \\
    &= \neg v^*(a) \Rightarrow (v^*(\neg b) \cup v^*((b \Rightarrow \neg c))) \\
    &= \neg v(a) \Rightarrow (\neg v(b) \cup (v(b) \Rightarrow \neg v(c))) \\
    &= \neg v_A(a) \Rightarrow (\neg v_A(b) \cup (v_A(b) \Rightarrow \neg v_A(c)))
\end{align*}
\]

\[
\begin{align*}
    (\neg T \Rightarrow (\neg T \cup (T \Rightarrow \neg F))) &= F \Rightarrow (F \cup T) = F \Rightarrow T = T, \text{ i.e.}
\end{align*}
\]

\[
\begin{align*}
    v_A &\models A \quad \text{and} \quad v \models A
\end{align*}
\]
Chapter 3: Question 2

Question 2
1. Find a restricted model and a restricted counter-model for \( A \), where

\[
A = (\neg a \Rightarrow (\neg b \cup (b \Rightarrow \neg c)))
\]

You can use short-hand notation. Show work

Solution

Notation: for any formula \( A \), we denote by \( \text{VAR}_A \) a set of all variables that appear in \( A \).

In our case we have \( \text{VAR}_A = \{a, b, c\} \).

Any function \( v_A : \text{VAR}_A \rightarrow \{T, F\} \) is called a truth assignment restricted to \( A \).

We define now \( v_A(a) = T, \quad v_A(b) = T, \quad v_A(c) = F \), in shorthand: \( a = T, \quad b = T, \quad c = F \) and evaluate

\[
(\neg T \Rightarrow (\neg T \cup (T \Rightarrow \neg F))) = F \Rightarrow (F \cup T) = F \Rightarrow T = T, \quad \text{i.e.}
\]

\[
v_A \models A
\]
Chapter 3: Question 2

Observe that

\( (\neg a \Rightarrow (\neg b \cup (b \Rightarrow \neg c))) = T \) when \( a = T \) and \( b, c \) any truth values as by definition of implication we have that

\( F \Rightarrow \text{anything} = T \)

Hence \( a = T \) gives us 4 models as we have \( 2^2 \) possible values on \( b \) and \( c \)
Chapter 3: Question 2

We take as a restricted counter-model: \( a = F, \ b = T \) and \( c = T \)

Evaluation: observe that

\[
(\neg a \Rightarrow (\neg b \cup (b \Rightarrow \neg c))) = F \quad \text{if and only if} \quad \\
\neg a = T \quad \text{and} \quad (\neg b \cup (b \Rightarrow \neg c)) = F \quad \text{if and only if} \quad \\
a = F, \ \neg b = F \quad \text{and} \quad (b \Rightarrow \neg c) = F \quad \text{if and only if} \quad \\
a = F, b = T \quad \text{and} \quad (T \Rightarrow \neg c) = F \quad \text{if and only if} \quad \\
a = F, b = T \quad \text{and} \quad \neg c = F \quad \text{if and only if} \quad \\
a = F, b = T \quad \text{and} \quad c = T
\]

The above proves also that \( a = F, \ b = T \) and \( c = T \) is the only restricted counter-model for \( A \)
Chapter 3: Question 3

Question 3  Justify whether the following statements true or false

S1  There are more then 3 possible restricted counter-models for A

S2  There are more then 2 possible restricted models of A

Solution

Statement:  There are more then 3 possible restricted counter-models for A is false

We have just proved that there is only one possible restricted counter-model for A

Statement:  There are more then 2 possible restricted models of A is true

There are 7 possible restricted models for A

Justification:  $2^3 - 1 = 7$
Chapter 3: Question 3

Question 3
1. List 3 models and 2 counter-models for $A$ from Question 2, i.e. for formula

$$A = (\neg a \Rightarrow (\neg b \cup (b \Rightarrow \neg c)))$$

that are extensions to the set $VAR$ of all variables of one the restricted models and of one of the restricted counter-models that you have found in Questions 1, 2
Chapter 3: Question 3

Solution

One of the restricted models is, for example a function $v_A : \{a, b, c\} \rightarrow \{T, F\}$ such that $v_A(a) = T$, $v_A(b) = T$, $v_A(c) = F$.

We extend $v_A$ to the set of all propositional variables $VAR$ to obtain a (non restricted) models as follows.
Model \( w_1 \) is a function
\[
w_1 : \text{VAR} \longrightarrow \{T, F\}
\]
such that
\[
w_1(a) = \nu_A(a) = T, \quad w_1(b) = \nu_A(b) = T,
\]
\[
w_1(c) = \nu_A(c) = F, \quad \text{and} \quad w_1(x) = T, \quad \text{for all}
\]
x \in \text{VAR} − \{a, b, c\}

Model \( w_2 \) is defined by a formula
\[
w_2(a) = \nu_A(a) = T, \quad w_2(b) = \nu_A(b) = T,
\]
\[
w_2(c) = \nu_A(c) = F, \quad \text{and} \quad w_2(x) = F, \quad \text{for all}
\]
x \in \text{VAR} − \{a, b, c\}
Chapter 3: Question 3

Model \( w_3 \) is defined by a formula
\[
w_3(a) = v_A(a) = T, \quad w_3(b) = v_A(b) = T, \quad w_3(c) = v(c) = F, \quad w_3(d) = F \quad \text{and} \quad w_3(x) = T \quad \text{for all} \quad x \in VAR - \{a, b, c, d\}
\]

There is as many of such models, as extensions of \( v_A \) to the set \( VAR \), i.e. as many as real numbers.
Chapter 3: Question 3

A counter-model for a formula $A = (\neg a \Rightarrow (\neg b \cup (b \Rightarrow \neg c)))$ is, by definition any function

$$v : \text{VAR} \longrightarrow \{T, F\}$$

such that $v^*(A) = F$

A restricted counter-model for the formula $A$, the only one, as already proved in is a function

$$v_A : \{a, b\} \longrightarrow \{T, F\}$$

such that

$$v_A(a) = F, \quad v_A(b) = T, \quad v_A(c) = T$$
Chapter 3: Question 3

We extend \( v_A \) to the set of all propositional variables \( VAR \) to obtain (non restricted) some counter-models. Here are two of such extensions

**Counter-model** \( w_1 \):

\[ w_1(a) = v_A(a) = F, \quad w_1(b) = v_A(b) = T, \]
\[ w_1(c) = v(c) = T, \quad \text{and} \quad w_1(x) = F, \quad \text{for all} \quad x \in VAR - \{a, b, c\} \]

**Counter-model** \( w_2 \):

\[ w_2(a) = v_A(a) = T, \quad w_2(b) = v_A(b) = T, \]
\[ w_2(c) = v(c) = T, \quad \text{and} \quad w_2(x) = T \quad \text{for all} \quad x \in VAR - \{a, b, c\} \]

There is as many of such counter-models, as extensions of \( v_A \) to the set \( VAR \), i.e. as many as real numbers.
Chapter 3: Models for Sets of Formulas

Definition
A truth assignment \( v \) is a **model for a set** \( G \subseteq \mathcal{F} \) of formulas of a given language \( \mathcal{L} = \mathcal{L}_{\{\neg, \Rightarrow, \cup, \cap\}} \) if and only if
\[
v \models B \quad \text{for all} \quad B \in G
\]
We denote it by \( v \models G \)

Observe that the set \( G \subseteq \mathcal{F} \) can be **finite** or **infinite**
Chapter 3: Consistent Sets of Formulas

Definition

A set $G \subseteq F$ of formulas is called consistent if and only if $G$ has a model, i.e. we have that $G \subseteq F$ is consistent if and only if there is $v$ such that $v \models G$

Otherwise $G$ is called inconsistent
Chapter 3: Independent Statements

Definition

A formula $A$ is called independent from a set $G \subseteq \mathcal{F}$ if and only if there are truth assignments $v_1, v_2$ such that

$v_1 \models G \cup \{A\}$ and $v_2 \models G \cup \{\neg A\}$

i.e. we say that a formula $A$ is independent if and only if

$G \cup \{A\}$ and $G \cup \{\neg A\}$ are consistent
Question 4
Given a set

\[ G = \{ ((a \cap b) \Rightarrow b), (a \cup b), \neg a \} \]

Show that \( G \) is **consistent**

**Solution**

We have to find \( v : VAR \rightarrow \{ T, F \} \) such that

\[ v \models G \]

It means that we need to find \( v \) such that

\[ v^*((a \cap b) \Rightarrow b) = T, \quad v^*(a \cup b) = T, \quad v^*(\neg a) = T \]
Chapter 3: Question 4

Observe that $\models ((a \land b) \Rightarrow b)$, hence we have that

1. $v^*((a \land b) \Rightarrow b) = T$ for any $v$
2. $v(a) = F$

$v^*(a \cup b) = v^*(a) \cup v^*(b) = v(a) \cup v(b) = F \cup v(b) = T$

only when $v(b) = T$ so we get

3. $v(b) = T$

This means that for any $v: \text{VAR} \rightarrow \{T, F\}$ such that $v(a) = F, \ v(b) = T, \ v \models G$

and we proved that $G$ is consistent
Chapter 3: Question 5

Question 5
Show that a formula \( A = (\neg a \land b) \) is not independent of
\[
G = \{ ((a \land b) \Rightarrow b), (a \lor b), \neg a \}
\]

Solution
We have to show that it is impossible to construct \( v_1, v_2 \) such that
\[
v_1 \models G \cup \{A\} \quad \text{and} \quad v_2 \models G \cup \{\neg A\}
\]

Observe that we have just proved that any \( v \) such that \( v(a) = F, \) and \( v(b) = T \) is the only model restricted to the set of variables \( \{a, b\} \) for \( G \) so we have to check now if it is possible that \( v \models A \) and \( v \models \neg A \)
Chapter 3: Question 5

We have to evaluate $v^*(A)$ and $v^*(\neg A)$ for $v(a) = F$, and $v(b) = T$
$v^*(A) = v^*((\neg a \land b)) = \neg v(a) \land v(b) = \neg F \land T = T \land T = T$
and so $v \models A$
$v^*(\neg A) = \neg v^*(A) = \neg T = F$
and so $v \not\models \neg A$

This ends the proof that $A$ is not independent of $G$
Question 6
Find an infinite number of formulas that are independent of

\[ G = \{ ((a \cap b) \Rightarrow b), (a \cup b), \neg a \} \]

This my solution - there are many others, but this one seemed to me to be the simplest

Solution
We just proved that any \( v \) such that \( v(a) = F, \ v(b) = T \) is the only model restricted to the set of variables \( \{a, b\} \) and so all other possible models for \( G \) must be extensions of \( v \)
Chapter 3: Question 6

We define a countably infinite set of formulas (and their negations) and corresponding extensions of \( \nu \) (restricted to to the set of variables \( \{a, b\} \)) such that \( \nu \models G \) as follows:

Observe that all extensions of \( \nu \) restricted to to the set of variables \( \{a, b\} \) have as domain the infinitely countable set

\[
\text{VAR} - \{a, b\} = \{a_1, a_2, \ldots, a_n, \ldots\}
\]

We take as a set of formulas (to be proved to be independent) the set of atomic formulas

\[
\mathcal{F}_0 = \text{VAR} - \{a, b\} = \{a_1, a_2, \ldots, a_n, \ldots\}
\]
proof of independence of any formula of $\mathcal{F}_0$

Let $c \in \mathcal{F}_0$

We define truth assignments $v_1, v_2 : \text{VAR} \rightarrow \{T, F\}$ such that

$v_1 \models \mathcal{G} \cup \{c\}$ and $v_2 \models \mathcal{G} \cup \{\neg c\}$

as follows

$v_1(a) = v(a) = F, \quad v_1(b) = v(b) = T$ and $v_1(c) = T$

for all $c \in \mathcal{F}_0$

$v_2(a) = v(a) = F, \quad v_2(b) = v(b) = T$ and $v_2(c) = F$

for all $c \in \mathcal{F}_0$
CHAPTER 3
Some Extensional Many Valued Semantics
Chapter 3: Question 7

Question 7
We define a 4 valued $H_4$ logic semantics as follows

The language is $\mathcal{L} = \mathcal{L}\{\neg, \Rightarrow, \lor, \land\}$

The logical connectives $\neg, \Rightarrow, \lor, \land$ of $H_4$ are operations in the set $\{F, \bot_1, \bot_2, T\}$, where $\{F < \bot_1 < \bot_2 < T\}$ and are defined as follows

Conjunction $\land$ is a function

$\land : \{F, \bot_1, \bot_2, T\} \times \{F, \bot_1, \bot_2, T\} \rightarrow \{F, \bot_1, \bot_2, T\}$, such that for any $x, y \in \{F, \bot_1, \bot_2, T\}$

$$x \land y = \text{min}\{x, y\}$$
Chapter 3: Question 7

**Disjunction**  \( \cup \) is a function
\[
\cup : \{F, \bot_1, \bot_2, T\} \times \{F, \bot_1, \bot_2, T\} \rightarrow \{F, \bot_1, \bot_2, T\},
\]
such that for any \( x, y \in \{F, \bot_1, \bot_2, T\} \)
\[
x \cup y = \max\{x, y\}
\]

**Implication**  \( \Rightarrow \) is a function
\[
\Rightarrow : \{F, \bot_1, \bot_2, T\} \times \{F, \bot_1, \bot_2, T\} \rightarrow \{F, \bot_1, \bot_2, T\},
\]
such that for any \( x, y \in \{F, \bot_1, \bot_2, T\}, \)
\[
x \Rightarrow y = \begin{cases} 
  T & \text{if } x \leq y \\
  y & \text{otherwise}
\end{cases}
\]

**Negation:**  for any \( x, y \in \{F, \bot_1, \bot_2, T\} \)
\[
\neg x = x \Rightarrow F
\]
Part 1   Write **Truth Tables** for **IMPLICATION** and **NEGATION** in $H_4$

**Solution**

$H_4$ Implication

<table>
<thead>
<tr>
<th>$\Rightarrow$</th>
<th>F</th>
<th>$\perp_1$</th>
<th>$\perp_2$</th>
<th>T</th>
</tr>
</thead>
<tbody>
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<td>T</td>
<td>T</td>
<td>T</td>
<td>T</td>
</tr>
<tr>
<td>$\perp_1$</td>
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<td>T</td>
<td>T</td>
<td>T</td>
</tr>
<tr>
<td>$\perp_2$</td>
<td>F</td>
<td>$\perp_1$</td>
<td>T</td>
<td>T</td>
</tr>
<tr>
<td>T</td>
<td>F</td>
<td>$\perp_1$</td>
<td>$\perp_2$</td>
<td>T</td>
</tr>
</tbody>
</table>

$H_4$ Negation

<table>
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<tr>
<th>$\neg$</th>
<th>F</th>
<th>$\perp_1$</th>
<th>$\perp_2$</th>
<th>T</th>
</tr>
</thead>
<tbody>
<tr>
<td>T</td>
<td>F</td>
<td>F</td>
<td>F</td>
<td>F</td>
</tr>
</tbody>
</table>
Part 2  Verify whether

\[ \models_{H_4} ((a \Rightarrow b) \Rightarrow (\neg a \cup b)) \]

Solution
Take any \( v \) such that
\( v(a) = \bot_1 \), \( v(b) = \bot_2 \)
Evaluate
\[ v^* ((a \Rightarrow b) \Rightarrow (\neg a \cup b)) = (\bot_1 \Rightarrow \bot_2) \Rightarrow (\neg \bot_1 \cup \bot_2) = T \Rightarrow (F \cup \bot_2)) = T \Rightarrow \bot_2 = \bot_2 \]
This proves that our \( v \) is a counter-model and hence

\[ \not\models_{H_4} ((a \Rightarrow b) \Rightarrow (\neg a \cup b)) \]
Chapter 3: Question 8

Question 8
Show that (can’t use TTables!)

\[ \models ((\neg a \cup b) \Rightarrow (((c \cap d) \Rightarrow \neg d) \Rightarrow (\neg a \cup b))) \]

Solution
Denote \( A = (\neg a \cup b) \), and \( B = ((c \cap d) \Rightarrow \neg d) \)

Our formula becomes a substitution of a **basic tautology**

\[ (A \Rightarrow (B \Rightarrow A)) \]

and hence is a **tautology**
Chapter 3: Challenge Exercise

1. Define your own propositional language $L_{CON}$ that contains also different connectives that the standard connectives $\neg$, $\cup$, $\cap$, $\Rightarrow$

Your language $L_{CON}$ does not need to include all (if any!) of the standard connectives $\neg$, $\cup$, $\cap$, $\Rightarrow$

2. Describe intuitive meaning of the new connectives of your language

3. Give some motivation for your own semantic

4. Define formally your own extensional semantics $M$ for your language $L_{CON}$ - it means

write carefully all Steps 1-4 of the definition of your $M$
Chapter 3: Question 9

Question 9

Definition

Let $S_3$ be a 3-valued semantics for $\mathcal{L}_{\{\neg, \cup, \Rightarrow\}}$ defined as follows:

$V = \{F, U, T\}$ is the set of logical values with the distinguished value $T$

$x \Rightarrow y = \neg x \cup y$ for any $x, y \in \{F, U, T\}$

$\neg F = T, \quad \neg U = F, \quad \neg T = U$

and

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Question 9

Part 1
Consider the following classical tautologies:

\[ A_1 = (a \cup \neg a), \quad A_2 = (a \Rightarrow (b \Rightarrow a)) \]

Find \( S_3 \) counter-models for \( A_1, A_2 \), if exist

You can’t use shorthand notation

Solution
Any \( v \) such that \( v(a) = v(b) = U \) is a counter-model for both \( A_1 \) and \( A_2 \), as

\[
\begin{align*}
    v^*(a \cup \neg a) &= v^*(a) \cup \neg v^*(b) = U \cup \neg U = U \cup F = U \neq T \\
    v^*(a \Rightarrow (b \Rightarrow a)) &= v^*(a) \Rightarrow (v^*(b) \Rightarrow v^*(a)) = U \Rightarrow (U \Rightarrow U) = U \Rightarrow U = \neg U \cup U = F \cup U = U \neq T
\end{align*}
\]
Question 9

Part 2
Consider the following classical tautologies:

\[ A_1 = (a \cup \neg a), \quad A_2 = (a \Rightarrow (b \Rightarrow a)) \]

Define your own 2-valued semantics \( S_2 \) for \( \mathcal{L} \), such that none of \( A_1, A_2 \) is a \( S_2 \) tautology
Verify your results. You can use shorthand notation.

Solution
This is not the only solution, but it is the simplest and most obvious I could think of! Here it is.
We define \( S_2 \) connectives as follows
\[ \neg x = F, \quad x \Rightarrow y = F, \quad x \cup y = F \] for all \( x, y \in \{F, T\} \)
Obviously, for any \( v \),
\[ v^*(a \cup \neg a) = F \] and \( v^*(a \Rightarrow (b \Rightarrow a)) = F \)
Chapter 3: Question 10

Question 10

Prove using proper **classical** logical equivalences (list them at each step) that for any formulas \( A, B \) of language \( \mathcal{L}_{\{\neg, \cup, \Rightarrow\}} \)

\[
\neg(A \iff B) \equiv ((A \cap \neg B) \cup (\neg A \cap B))
\]

Solution

\[
\neg(A \iff B) \equiv^{\text{def}} \neg((A \Rightarrow B) \cap (B \Rightarrow A))
\]

\[
\equiv^{\text{deMorgan}} (\neg(A \Rightarrow B) \cup \neg(B \Rightarrow A))
\]

\[
\equiv^{\text{negimpl}} ((A \cap \neg B) \cup (B \cap \neg A)) \equiv^{\text{commut}} ((A \cap \neg B) \cup (\neg A \cap B))
\]
Question 11

Question 11
Prove using proper **classical** logical equivalences (list them at each step) that for any formulas $A, B$ of language $\mathcal{L}_{\{\neg, \cup, \Rightarrow\}}$

$$((B \cap \neg C) \Rightarrow (\neg A \cup B)) \equiv ((B \Rightarrow C) \cup (A \Rightarrow B))$$

**Solution**

$$((B \cap \neg C) \Rightarrow (\neg A \cup B))$$

$\equiv^{\text{impl}}(\neg (B \cap \neg C) \cup (\neg A \cup B))$

$\equiv^{\text{deMorgan}}((\neg B \cup \neg \neg C) \cup (\neg A \cup B))$

$\equiv^{\text{dneg}}((\neg B \cup C) \cup (\neg A \cup B)) \equiv^{\text{impl}}((B \Rightarrow C) \cup (A \Rightarrow B))$
We define Ł connectives for $\mathcal{L}\{\neg, \cup, \Rightarrow\}$ as follows:

**Ł Negation** $\neg$ is a function:

$$\neg : \{T, \bot, F\} \rightarrow \{T, \bot, F\}$$

such that $\neg \bot = \bot$, $\neg T = F$, $\neg F = T$

**Ł Conjunction** $\cap$ is a function:

$$\cap : \{T, \bot, F\} \times \{T, \bot, F\} \rightarrow \{T, \bot, F\}$$

such that $x \cap y = \min\{x, y\}$ for all $x, y \in \{T, \bot, F\}$

Remember that we assumed: $F < \bot < T$
Ł Implication ⇒ is a function:

⇒: \{T, \bot, F\} \times \{T, \bot, F\} \rightarrow \{T, \bot, F\}

such that

\[ x \Rightarrow y = \begin{cases} \neg x \cup y & \text{if } x > y \\ T & \text{otherwise} \end{cases} \]

Given a formula \(((a \cap b) \Rightarrow \neg b) \in \mathcal{F}\) of \(\mathcal{L}_{\neg, \cup, \Rightarrow}\)

Use the fact that \(v: \text{VAR} \rightarrow \{F, \bot, T\}\) is such that

\(v^*(((a \cap b) \Rightarrow \neg b)) = \bot\) under Ł semantics to evaluate

\(v^*(((b \Rightarrow \neg a) \Rightarrow (a \Rightarrow \neg b)) \cup (a \Rightarrow b))\)

You can use shorthand notation
Solution

The formula \(((a \cap b) \Rightarrow \neg b) = \bot\) in Ł connectives semantics in two cases written is the shorthand notation as

C1 \((a \cap b) = \bot\) and \(\neg b = F\)

C2 \((a \cap b) = T\) and \(\neg b = \bot\).

Consider case C1

\(\neg b = F\), so \(v(b) = T\), and hence \((a \cap T) = v(a) \cap T = \bot\)

if and only if \(v(a) = \bot\)

It means that \(v^*(((a \cap b) \Rightarrow \neg b)) = \bot\) for any \(v\), is such that \(v(a) = \bot\) and \(v(b) = T\)
We now evaluate (in shorthand notation)

\[ v^*(((b \rightarrow \neg a) \rightarrow (a \rightarrow \neg b)) \cup (a \rightarrow b)) \]

\[ = (((T \rightarrow \neg \bot) \rightarrow (\bot \rightarrow \neg T)) \cup (\bot \rightarrow T)) = ((\bot \rightarrow \bot) \cup T) = T \]

Consider now Case C2

\( \neg b = \bot \), i.e. \( b = \bot \), and hence \( (a \cap \bot) = T \) what is impossible, hence \( v \) from the Case C1 is the only one
Question 13

Use the **Definability of Conjunction** in terms of disjunction and negation **Equivalence**

\[(A \cap B) \equiv \neg(\neg A \cup \neg B)\]

to transform a formula

\[A = \neg(\neg(\neg a \cap \neg b) \cap a)\]

of the language \(L_{\{\cap,\neg}\}\) into a logically equivalent formula \(B\) of the language \(L_{\{\cup,\neg}\}\)
Question 13

Solution

\[\neg(\neg(a \cap \neg b) \cap a) \equiv \neg(\neg(\neg a \cap \neg b) \cup \neg a)\]

\[\equiv ((\neg a \cap \neg b) \cup \neg a) \equiv (\neg(\neg a \cup \neg b) \cup \neg a)\]

\[\equiv \neg(a \cup b) \cup \neg a\]

The formula $B$ of $L_{\{\cup, \neg\}}$ equivalent to $A$ is

$B = (\neg(a \cup b) \cup \neg a)$
Equivalence of Languages Definition

Definition
Given two languages: \( L_1 = L_{\text{CON}_1} \) and \( L_2 = L_{\text{CON}_2} \), for \( \text{CON}_1 \neq \text{CON}_2 \)
We say that they are logically equivalent, i.e.

\[
L_1 \equiv L_2
\]

if and only if the following conditions \( \textbf{C1}, \textbf{C2} \) hold.

\( \textbf{C1} \): for any formula \( A \) of \( L_1 \), there is a formula \( B \) of \( L_2 \), such that \( A \equiv B \)

\( \textbf{C2} \): for any formula \( C \) of \( L_2 \), there is a formula \( D \) of \( L_1 \), such that \( C \equiv D \)
Question 15

Prove the logical equivalence of the languages

\[ L\{\neg, \cup\} \equiv L\{\neg, \Rightarrow\} \]

Solution
We need two definability equivalences:
implication in terms of disjunction and negation

\[ (A \Rightarrow B) \equiv (\neg A \cup B) \]

and disjunction in terms of implication and negation,

\[ (A \cup B) \equiv (\neg A \Rightarrow B) \]

and the Substitution Theorem
Question 16

Prove the logical equivalence of the languages

\[ \mathcal{L}_{\neg, \cap, \cup, \Rightarrow} \equiv \mathcal{L}_{\neg, \cap, \cup} \]

Solution

We need only the **definability of implication** in terms of disjunction and negation equivalence

\[ (A \Rightarrow B) \equiv (\neg A \cup B) \]

as the **Substitution Theorem** for any formula \( A \) of \( \mathcal{L}_{\neg, \cap, \cup, \Rightarrow} \) there is a formula \( B \) of \( \mathcal{L}_{\neg, \cap, \cup} \) such that \( A \equiv B \) and the condition \( C_1 \) holds

Observe that any formula \( A \) of language \( \mathcal{L}_{\neg, \cap, \cup} \) is also a formula of the language \( \mathcal{L}_{\neg, \cap, \cup, \Rightarrow} \) and of course \( A \equiv A \) so the condition \( C_2 \) also holds