cse371/math371
LOGIC

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LECTURE 3e

Chapter 3 REVIEW
Some Definitions and Problems

SOME DEFINITIONS: Part One

There are some basic **Definitions** and sample **Questions** with Solutions from Chapter 3

Study them them for MIDTERM

Knowing all basic **Definitions** is the first step for understanding the material and solve Problems

Solutions are very carefully written - so you could understand them step by step and hence correctly write yours, which do not need to be that detailed

DEFINITIONS: Propositional Extensional Semantics

Definition 1

Given a propositional language $\mathcal{L}_{CON}$ for the set $CON = C_1 \cup C_2$, where C_1, C_2 are respectively the sets of unary and binary connectives

Let V be a non-empty set of **logical values**

Connectives $\triangledown \in C_1$, $\circ \in C_2$ are called **extensional** iff their semantics is defined by respective functions

$$\triangledown : \; V \longrightarrow V \quad \text{and} \quad \circ : \; V \times V \longrightarrow V$$

DEFINITIONS: Propositional Extensional Semantics

Definition 2

Formal definition of a **propositional extensional semantics** for a given language $\mathcal{L}_{CON}$ consists of providing **definitions** of the following four main components:

1. Logical Connectives
2. Truth Assignment
3. Satisfaction, Model, Counter-Model
4. Tautology

CLASSICAL PROPOSITIONAL SEMANTICS

DEFINITIONS: Truth Assignment Extension v^*

Definition 3

The Language: $\mathcal{L} = \mathcal{L}_{\{\neg, \Rightarrow, \cup, \cap\}}$

Given the truth assignment $v : VAR \longrightarrow \{T, F\}$ in **classical semantics** for the language $\mathcal{L} = \mathcal{L}_{\{\neg, \Rightarrow, \cup, \cap\}}$

We define its **extension** v^* to the set $\mathcal{F}$ of all formulas of $\mathcal{L}$ as $v^* : \mathcal{F} \longrightarrow \{T, F\}$ such that

(i) for any $a \in VAR$

$$v^*(a) = v(a)$$

(ii) and for any $A, B \in \mathcal{F}$ we put

$$v^*(\neg A) = \neg v^*(A);$$

$$v^*((A \cap B)) = \cap(v^*(A), v^*(B));$$

$$v^*((A \cup B)) = \cup(v^*(A), v^*(B));$$

$$v^*((A \Rightarrow B)) = \Rightarrow(v^*(A), v^*(B));$$

$$v^*((A \Leftrightarrow B)) = \Leftrightarrow(v^*(A), v^*(B))$$

DEFINITIONS: Truth Assignment Extension v^* Revisited

Notation

For binary connectives (two argument functions) we adopt a convention to write the symbol of the connective (name of the 2 argument function) between its arguments as we do in a case arithmetic operations

The **condition (ii)** of the definition of the extension v^* can be hence **written** as follows

(ii) and for any $A, B \in \mathcal{F}$ we put

$$v^*(\neg A) = \neg v^*(A);$$

$$v^*((A \cap B)) = v^*(A) \cap v^*(B);$$

$$v^*((A \cup B)) = v^*(A) \cup v^*(B);$$

$$v^*((A \Rightarrow B)) = v^*(A) \Rightarrow v^*(B);$$

$$v^*((A \Leftrightarrow B)) = v^*(A) \Leftrightarrow v^*(B)$$

DEFINITIONS: Satisfaction Relation

Definition 4 Let $v : VAR \longrightarrow \{T, F\}$

We say that

v **satisfies** a formula $A \in \mathcal{F}$ iff $v^*(A) = T$

Notation: $v \models A$

We say that

v **does not satisfy** a formula $A \in \mathcal{F}$ iff $v^*(A) \neq T$

Notation: $v \not\models A$

DEFINITIONS: Model, Counter-Model, Classical Tautology

Definition 5

Given a formula $A \in \mathcal{F}$ and $v : VAR \longrightarrow \{T, F\}$

We say that

v is a **model** for A iff $v \models A$

v is a **counter-model** for A iff $v \not\models A$

Definition 6

A is a **tautology** iff for any $v : VAR \longrightarrow \{T, F\}$ we have that $v \models A$

Notation

We write symbolically $\models A$ to denote that A is a **classical tautology**

DEFINITIONS: Restricted Truth Assignments

Notation: for any formula A, we denote by VAR_A a set of all variables that appear in A

Definition 7 Given a formula $A \in \mathcal{F}$, any function

$$v_A : \quad VAR_A \longrightarrow \{T, F\}$$

is called a truth assignment restricted to A

DEFINITIONS: Restricted Model, Counter Model

Notation: for any formula A, we denote by VAR_A a set of all variables that appear in A

Definition 8 Given a formula $A \in \mathcal{F}$

Any function

$$w : \quad VAR_A \longrightarrow \{T, F\} \quad \text{such that} \quad w^*(A) = T$$

is called a **restricted MODEL** for A

Any function

$$w : \quad VAR_A \longrightarrow \{T, F\} \quad \text{such that} \quad w^*(A) \neq T$$

is called a **restricted Counter- MODEL** for A

DEFINITIONS: Models for Sets of Formulas

Consider $\mathcal{L} = \mathcal{L}_{\{\neg, \cup, \cap, \Rightarrow\}}$ and let $\mathcal{S} \neq \emptyset$ be any non empty set of formulas of $\mathcal{L}$, i.e.

$$\mathcal{S} \subseteq \mathcal{F}$$

Definition 9

A truth truth assignment $v : VAR \longrightarrow \{T, F\}$

is a **model for the set** $\mathcal{S}$ of formulas if and only if

$$v \models A \quad \textbf{for all formulas} \quad A \in \mathcal{S}$$

We write

$$v \models \mathcal{S}$$

to denote that v **is a model for the set** $\mathcal{S}$ of formulas

DEFINITIONS: Consistent Sets of Formulas

Definition 10

A non-empty set $\mathcal{G} \subseteq \mathcal{F}$ of **formulas** is called **consistent** if and only if $\mathcal{G}$ **has a model**, i.e. we have that

$\mathcal{G} \subseteq \mathcal{F}$ is **consistent** if and only if **there is** v such that $v \models \mathcal{G}$

Otherwise $\mathcal{G}$ is called **inconsistent**

DEFINITIONS: Independent Statements

Definition 11

A formula A is called **independent** from a non-empty set $\mathcal{G} \subseteq \mathcal{F}$

if and only if **there are** truth assignments v_1, v_2 such that

$$v_1 \models \mathcal{G} \cup \{A\} \quad \text{and} \quad v_2 \models \mathcal{G} \cup \{\neg A\}$$

i.e. we say that a formula A is **independent**

if and only if

$$\mathcal{G} \cup \{A\} \quad \text{and} \quad \mathcal{G} \cup \{\neg A\} \quad \text{are } \textbf{consistent}$$

Many Valued Extensional Semantics **M**

DEFINITIONS: Semantics **M**

Definition 11

The extensional semantics **M** is defined for a non-empty set of V of **logical values of any cardinality**

We only **assume** that the set V of logical values of **M** always has a special, distinguished logical value which serves to define a notion of tautology

We denote this distinguished value as T

Formal definition of **many valued extensional semantics M** for the language $\mathcal{L}_{CON}$ consists of giving **definitions** of the following main components:

1. Logical Connectives under semantics **M**
2. Truth Assignment for **M**
3. Satisfaction Relation, Model, Counter-Model under semantics **M**
4. Tautology under semantics **M**

Definition of **M** - Extensional Connectives

Given a propositional language $\mathcal{L}_{CON}$ for the set $CON = C_1 \cup C_2$, where C_1 is the set of all unary connectives, and C_2 is the set of all binary connectives

Let V be a non-empty set of **logical values** adopted by the semantics **M**

Definition 12

Connectives $\triangledown \in C_1$, $\circ \in C_2$ are called **M -extensional** iff their semantics **M** is defined by respective functions

$$\triangledown : \ V \longrightarrow V \quad \text{and} \quad \circ : \ V \times V \longrightarrow V$$

DEFINITION: Definability of Connectives under a semantics **M**

Given a propositional language $\mathcal{L}_{CON}$ and its **extensional semantics M**

We adopt the following definition

Definition 13

A connective $\circ \in CON$ is **definable** in terms of some connectives $\circ_1, \circ_2, ...\circ_n \in CON$ for $n \geq 1$ **under the semantics M** if and only if the connective $\circ$ is a certain function composition of functions $\circ_1, \circ_2, ...\circ_n$ as they are **defined by the semantics M**

DEFINITION: **M** Truth Assignment Extension v^* to $\mathcal{F}$

Definition 14

Given the **M** truth assignment $v : VAR \longrightarrow V$

We define its **M extension** v^* to the set $\mathcal{F}$ of all formulas of $\mathcal{L}$ as any function $v^* : \mathcal{F} \longrightarrow V$, such that the following conditions are satisfied

(i) for any $a \in VAR$

$$v^*(a) = v(a);$$

(ii) For any connectives $\triangledown \in C_1$, $\circ \in C_2$ and for any formulas $A, B \in \mathcal{F}$ we put

$$v^*(\triangledown A) = \triangledown v^*(A)$$

$$v^*((A \circ B)) = \circ(v^*(A), v^*(B))$$

DEFINITION: **M** Satisfaction, Model, Counter Model, Tautology

Definition 15 Let $v : VAR \longrightarrow V$

Let $T \in V$ be the **distinguished logical value**

We say that

v **M** satisfies a formula $A \in \mathcal{F}$ ($v \models_{\mathbf{M}} A$) iff $v^*(A) = T$

Definition 16

Given a formula $A \in \mathcal{F}$ and $v : VAR \longrightarrow V$

Any v such that $v \models_{\mathbf{M}} A$ is called a **M model** for A

Any v such that $v \not\models_{\mathbf{M}} A$ is called a **M counter model** for A

A is a **M tautology** ($\models_{\mathbf{M}} A$) iff $v \models_{\mathbf{M}} A$, **for all** $v : VAR \longrightarrow V$

CHAPTER 3: Some Sample Questions with Solutions

Question 1

Find a restricted model for formula A, where

$$A = (\neg a \Rightarrow (\neg b \cup (b \Rightarrow \neg c)))$$

You **can't use** short-hand notation

Show each step of solution

Solution

For any formula A, we denote by VAR_A a set of all variables that appear in A

In our case we have $VAR_A = \{a, b, c\}$

Any function $v_A : VAR_A \longrightarrow \{T, F\}$ is called a truth assignment restricted to A

Chapter 3: Question 1

Let $v : VAR \longrightarrow \{T, F\}$ be any truth assignment such that

$$v(a) = v_A(a) = T,\ v(b) = v_A(b) = T,\ v(c) = v_A(c) = F$$

We evaluate the value of the **extension** v^* of v on the formula A as follows

$v^*(A) = v^*((\neg a \Rightarrow (\neg b \cup (b \Rightarrow \neg c))))$

$= v^*(\neg a) \Rightarrow v^*((\neg b \cup (b \Rightarrow \neg c)))$

$= \neg v^*(a) \Rightarrow (v^*(\neg b) \cup v^*((b \Rightarrow \neg c)))$

$= \neg v(a) \Rightarrow (\neg v(b) \cup (v(b) \Rightarrow \neg v(c)\ ))$

$= \neg v_A(a) \Rightarrow (\neg v_A(b) \cup (v_A(b) \Rightarrow \neg v_A(c)\ ))$

$(\neg T \Rightarrow (\neg T \cup (T \Rightarrow \neg F))) = F \Rightarrow (F \cup T) = F \Rightarrow T = T$, i.e.

$$v_A \models A \quad \text{and} \quad v \models A$$

Question 2

Find a restricted model and a restricted counter-model for A, where

$$A = (\neg a \Rightarrow (\neg b \cup (b \Rightarrow \neg c)))$$

You **can use** short-hand notation. Show work

Solution

Notation: for any formula A, we denote by VAR_A a set of all variables that appear in A

In our case we have $VAR_A = \{a, b, c\}$

Any function $v_A : VAR_A \longrightarrow \{T, F\}$ is called a truth assignment restricted to A

We define now $v_A(a) = T,\ v_A(b) = T,\ v_A(c) = F$, in shorthand: $a = T, b = T, c = F$ and evaluate

$(\neg T \Rightarrow (\neg T \cup (T \Rightarrow \neg F))) = F \Rightarrow (F \cup T) = F \Rightarrow T = T$, i.e.

$$v_A \models A$$

Chapter 3: Question 2

Observe that

$(\neg a \Rightarrow (\neg b \cup (b \Rightarrow \neg c)) = T$ when $a = T$ and b, c any truth values as by definition of implication we have that

$F \Rightarrow \text{anything} = T$

Hence $a = T$ gives us 4 models as we have 2^2 possible values on b and c

Chapter 3: Question 2

We take as a restricted counter-model: a=F, b =T and c=T

Evaluation: observe that

$(\neg a \Rightarrow (\neg b \cup (b \Rightarrow \neg c)) = F$ if and only if

$\neg a = T$ and $(\neg b \cup (b \Rightarrow \neg c)) = F$ if and only if

$a = F$, $\neg b = F$ and $(b \Rightarrow \neg c) = F$ if and only if

$a = F, b = T$ and $(T \Rightarrow \neg c) = F$ if and only if

$a = F, b = T$ and $\neg c = F$ if and only if

$a = F, b = T$ and $c = T$

The above proves also that a=F, b =T and c=T is the only **restricted counter -model** for A

Question 3 Justify whether the following statements **true** or **false**

S1 There are more then 3 possible restricted counter-models for A

S2 There are more then 2 possible restricted models of A

Solution

S1Statement: There are more then 3 possible restricted counter-models for A is **false**

We have just proved that there is only one possible restricted counter-model for A

S2 Statement: There are more then 2 possible restricted models of A is **true**

There are 7 possible restricted models for A

Justification: $2^3 - 1 = 7$

Question 4

1. List **3 models** for A from **Question 2**, i.e. for formula

$$A = (\neg a \Rightarrow (\neg b \cup (b \Rightarrow \neg c)))$$

that are **extensions** to the set VAR of all variables of **one** of the restricted models that you have found in Questions 1,

2. List **2 counter models** for A that are **extensions** of **one** of the restricted countrer models that you have found in the Questions 1, 2

Chapter 3: Question 4

Solution

1. One of the **restricted models** is, for example a function

$v_A : \{a, b, c\} \longrightarrow \{T, F\}$ such that

$v_A(a) = T, \; v_A(b) = T, \; v_A(c) = F$

We **extend** v_A to the set of all propositional variables VAR to obtain a (non restricted) **models** as follows

Chapter 3: Question 4

Model w_1 is a function

$w_1 : VAR \longrightarrow \{T, F\}$ such that

$w_1(a) = v_A(a) = T$, $w_1(b) = v_A(b) = T$,

$w_1(c) = v_A(c) = F$, and $w_1(x) = T$, for all

$x \in VAR - \{a, b, c\}$

Model w_2 is defined by a formula

$w_2(a) = v_A(a) = T$, $w_2(b) = v_A(b) = T$,

$w_2(c) = v_A(c) = F$, and $w_2(x) = F$, for all

$x \in VAR - \{a, b, c\}$

Chapter 3: Question 4

Model w_3 is defined by a formula

$w_3(a) = v_A(a) = T,\ w_3(b) = v_A(b) = T,\ w_3(c) = v(c) = F,$
$w_3(d) = F$ and $w_3(x) = T$ for all $x \in VAR - \{a, b, c, d\}$

There is as many of such models, as extensions of v_A to the set VAR, i.e. as many as real numbers

2. A **counter-model** for a formula

$A = (\neg a \Rightarrow (\neg b \cup (b \Rightarrow \neg c))$ is, by **definition** any function

$$v : VAR \longrightarrow \{T, F\}$$

such that $v^*(A) = F$

A restricted counter-model for the formula A, the only one, as already proved in is a function

$$v_A : \{a, b\} \longrightarrow \{T, F\}$$

such that such that

$$v_A(a) = F, \ \ v_A(b) = T, \ \ v_A(c) = T$$

Chapter 3: Question 4

We extend v_A to the set of all propositional variables VAR to obtain (non restricted) some counter-models.

Here are **two** of such extensions

Counter- model w_1:

$w_1(a) = v_A(a) = F$, $w_1(b) = v_A(b) = T$,
$w_1(c) = v(c) = T$, and $w_1(x) = F$, for all $x \in VAR - \{a, b, c\}$

Counter- model w_2:

$w_2(a) = v_A(a) = T$, $w_2(b) = v_A(b) = T$,
$w_2(c) = v(c) = T$, and $w_2(x) = T$ for all $x \in VAR - \{a, b, c\}$

There is as many of such **counter- models**, as extensions of v_A to the set VAR, i.e. as many as real numbers

Chapter 3: Models for Sets of Formulas

Definition

A truth assignment v is a **model for a set** $\mathcal{G} \subseteq \mathcal{F}$ **of formulas** of a given language $\mathcal{L} = \mathcal{L}_{\{\neg, \Rightarrow, \cup, \cap\}}$ if and only if

$$v \models B \quad \textbf{for all} \quad B \in \mathcal{G}$$

We denote it by $v \models \mathcal{G}$

Observe that the set $\mathcal{G} \subseteq \mathcal{F}$ can be **finite** or **infinite**

Chapter 3: Consistent Sets of Formulas

Definition

A set $\mathcal{G} \subseteq \mathcal{F}$ of **formulas** is called **consistent**

if and only if $\mathcal{G}$ **has a model**, i.e. we have that

$\mathcal{G} \subseteq \mathcal{F}$ is **consistent** if and only if

there is v such that $v \models \mathcal{G}$

Otherwise $\mathcal{G}$ is called **inconsistent**

Chapter 3: Independent Statements

Definition

A formula A is called **independent** from a set $\mathcal{G} \subseteq \mathcal{F}$ if and only if **there are** truth assignments v_1, v_2 such that

$$v_1 \models \mathcal{G} \cup \{A\} \quad \text{and} \quad v_2 \models \mathcal{G} \cup \{\neg A\}$$

i.e. we say that a formula A is **independent** if and only if

$$\mathcal{G} \cup \{A\} \quad \text{and} \quad \mathcal{G} \cup \{\neg A\} \quad \text{are } \textbf{consistent}$$

Question 5

Given a set $\mathcal{G} = \{((a \cap b) \Rightarrow b),\ (a \cup b), \neg a\}$

Show that $\mathcal{G}$ is **consistent**

Solution

We have to find $v : VAR \longrightarrow \{T, F\}$ such that $v \models \mathcal{G}$

It means that we need to **find** a v such that

$$v^*((a \cap b) \Rightarrow b) = T, \quad v^*(a \cup b) = T, \quad v^*(\neg a) = T$$

We write it in the **shorthand notation**

$$((a \cap b) \Rightarrow b) = T, \quad (a \cup b) = T, \quad \neg a = T$$

We have to find out of it is possible

Chapter 3: Question 5

1. Observe that $\models ((a \cap b) \Rightarrow b)$, hence we have that $v^*((a \cap b) \Rightarrow b) = T$ for any v
2. Case $\neg a = T$ holds if and only if $a = F$
3. Case $(a \cup b) = T$ holds if and only if $(T \cup b) = T$ as $a = F$, and this holds if and only if $b = T$

   This **proves** that for any $v : VAR \longrightarrow \{T, F\}$ such that $v(a) = F$, $v(b) = T$, is a **model** for $\mathcal{G}$ and so, by definition, that $\mathcal{G}$ is **consistent**

   Moreover, we have **proved** that it is the **only** (restricted) model for $\mathcal{G}$

Question 6

Show that a formula $A = (\neg a \cap b)$ is **not independent** of

$$\mathcal{G} = \{((a \cap b) \Rightarrow b),\ (a \cup b), \neg a\}$$

Solution

We have to show that **it is impossible** to construct v_1, v_2 such that

$$v_1 \models \mathcal{G} \cup \{A\} \text{ and } v_2 \models \mathcal{G} \cup \{\neg A\}$$

Observe that we have just proved that any v such that $v(a) = F,$ and $v(b) = T$ is **the only** model restricted to the set of variables $\{a, b\}$ for $\mathcal{G}$ so we have to check now if it is **possible** that for that formula $A = (\neg a \cap b),$ $v \models A$ and $v \models \neg A$

Chapter 3: Question 6

We have to evaluate $v^*(A)$ and $v^*(\neg A)$ for

$v(a) = F$, and $v(b) = T$

$v^*(A) = v^*((\neg a \cap b) = \neg v(a) \cap v(b) = \neg F \cap T = T \cap T = T$

and so $v \models A$

$v^*(\neg A) = \neg v^*(A) = \neg T = F$

and so $v \not\models \neg A$

This **ends the proof** that A is **not independent** of $\mathcal{G}$

Question 7

Find an infinite number of formulas that are **independent** of

$$\mathcal{G} = \{((a \cap b) \Rightarrow b),\ (a \cup b), \neg a\}$$

This **my solution** - there are many others, but this one seemed to me to be the **simplest**

Solution

We just proved that any v such that $v(a) = F, \ \ v(b) = T$ is **the only** model restricted to the set of variables $\{a, b\}$ and so all other possible models for $\mathcal{G}$ must be **extensions** of v

Chapter 3: Question 7

We **define** a countably infinite set of formulas (and their negations) and corresponding **extensions** of v (restricted to to the set of variables $\{a, b\}$) such that $v \models \mathcal{G}$ as follows

Observe that **all extensions** of v restricted to to the set of variables $\{a, b\}$ have as domain the infinitely countable set

$$VAR - \{a,\ b\} = \{a_1,\ a_2,\ \ldots, a_n, \ldots\}$$

We take as a set of formulas (to be **proved** to be independent) the set of **atomic formulas**

$$\mathcal{F}_0 = VAR - \{a,\ b\} = \{a_1,\ a_2,\ \ldots, a_n, \ldots\}$$

Chapter 3: Question 7

proof of independence of any formula of $\mathcal{F}_0$

Let $c \in \mathcal{F}_0$

We define truth assignments $v_1,\ v_2 : VAR \longrightarrow \{T, F\}$

such that

$$v_1 \models \mathcal{G} \cup \{c\} \ \text{ and } \ v_2 \models \mathcal{G} \cup \{\neg c\}$$

as follows

$v_1(a) = v(a) = F, \quad v_1(b) = v(b) = T$ and $v_1(c) = T$

for all $c \in \mathcal{F}_0$

$v_2(a) = v(a) = F, \quad v_2(b) = v(b) = T$ and $v_2(c) = F$

for all $c \in \mathcal{F}_0$

CHAPTER 3
Some Extensional Many Valued Semantics

Question 8

We **define** a 4 valued $\mathbf{H}_4$ logic semantics as follows

The language is $\mathcal{L} = \mathcal{L}_{\{\neg, \Rightarrow, \cup, \cap\}}$

The logical connectives $\neg, \Rightarrow, \cup, \cap$ of $\mathbf{H}_4$ are operations in the set $\{F, \bot_1, \bot_2, T\}$, where $\{F < \bot_1 < \bot_2 < T\}$ and are defined as follows

Conjunction $\cap$ is a function

$\cap : \ \{F, \bot_1, \bot_2, T\} \times \{F, \bot_1, \bot_2, T\} \longrightarrow \{F, \ \bot_1, \bot_2, , T\},$

such that for any $x, y \in \{F, \bot_1, \bot_2, T\}$

$$x \cap y = min\{x, y\}$$

Chapter 3: Question 8

Disjunction $\cup$ is a function

$\cup$: $\{F, \perp_1, \perp_2, T\} \times \{F, \perp_1, \perp_2, T\} \longrightarrow \{F, \perp_1, \perp_2, T\}$,

such that for any $x, y \in \{F, \perp_1, \perp_2, T\}$

$$x \cup y = max\{x, y\}$$

Implication $\Rightarrow$ is a function

$\Rightarrow$: $\{F, \perp_1, \perp_2, T\} \times \{F, \perp_1, \perp_2, T\} \longrightarrow \{F, \perp_1, \perp_2, T\}$,

such that for any $x, y \in \{F, \perp_1, \perp_2, T\}$,

$$x \Rightarrow y = \begin{cases} T & \text{if } x \leq y \\ y & \text{otherwise} \end{cases}$$

Negation: for any $x, y \in \{F, \perp_1, \perp_2, T\}$

$$\neg x = x \Rightarrow F$$

Chapter 3: Question 8

Part 1 Write Truth Tables for IMPLICATION and NEGATION in $\mathbf{H_4}$

Solution

H_4 Implication

$\Rightarrow$	F	$\perp_1$	$\perp_2$	T
F	T	T	T	T
$\perp_1$	F	T	T	T
$\perp_2$	F	$\perp_1$	T	T
T	F	$\perp_1$	$\perp_2$	T

H_4 Negation

$\neg$	F	$\perp_1$	$\perp_2$	T
	T	F	F	F

Part 2 Verify whether

$$\models_{\mathbf{H}_4} ((a \Rightarrow b) \Rightarrow (\neg a \cup b))$$

Solution

Take any v such that

$v(a) = \bot_1 \quad v(b) = \bot_2$

Evaluate

$v * ((a \Rightarrow b) \Rightarrow (\neg a \cup b)) = (\bot_1 \Rightarrow \bot_2) \Rightarrow (\neg \bot_1 \cup \bot_2) =$
$T \Rightarrow (F \cup \bot_2)) = T \Rightarrow \bot_2 = \bot_2$

This proves that our v is a **counter-model** and hence

$$\not\models_{\mathbf{H}_4} ((a \Rightarrow b) \Rightarrow (\neg a \cup b))$$

Question 9

Show that (can't use TTables!)

$$\models ((\neg a \cup b) \Rightarrow (((c \cap d) \Rightarrow \neg d) \Rightarrow (\neg a \cup b)))$$

Solution

Denote $A = (\neg a \cup b)$, and $B = ((c \cap d) \Rightarrow \neg d)$

Our formula becomes a substitution of a **basic tautology**

$$(A \Rightarrow (B \Rightarrow A))$$

and hence is a **tautology**

Chapter 3: Challenge Exercise

1. Define your own propositional language $\mathcal{L}_{CON}$ that contains also **different connectives** that the standard connectives $\neg, \cup, \cap, \Rightarrow$

Your language $\mathcal{L}_{CON}$ **does not need** to include all (if any!) of the standard connectives $\neg, \cup, \cap, \Rightarrow$

2. Describe intuitive meaning of the new connectives of your language

3. Give some motivation for **your own semantic**

4. Define formally your own **extensional semantics M** for your language $\mathcal{L}_{CON}$ - it means

write carefully all **Steps 1- 4** of the definition of your **M**

Question 10

Definition

Let S_3 be a 3-valued semantics for $\mathcal{L}_{\{\neg,\, \cup,\, \Rightarrow\}}$ defined as follows:

$V = \{F, U, T\}$ is the set of logical values with the distinguished value T

$x \Rightarrow y = \neg x \cup y$ for any $x, y \in \{F, U, T\}$

$$\neg F = T, \quad \neg U = F, \quad \neg T = U$$

and

$\cup$	F	U	T
F	F	U	T
U	U	U	U
T	T	U	T

Question 10

Part 1

Consider the following classical tautologies:

$$A_1 = (a \cup \neg a), \qquad A_2 = (a \Rightarrow (b \Rightarrow a))$$

Find S_3 **counter-models** for A_1, A_2, if exist

You **can't use shorthand** notation

Solution

Any v such that $v(a) = v(b) = U$ is a **counter-model** for both A_1 and A_2, as

$v^*(a \cup \neg a) = v^*(a) \cup \neg v^*(b) = U \cup \neg U = U \cup F = U \neq T$

$v^*(a \Rightarrow (b \Rightarrow a)) = v^*(a) \Rightarrow (v^*(b) \Rightarrow v^*(a)) = U \Rightarrow (U \Rightarrow U) = U \Rightarrow U = \neg U \cup U = F \cup U = U \neq T$

Question 10

Part 2

Consider the following classical tautologies:

$$A_1 = (a \cup \neg a), \qquad A_2 = (a \Rightarrow (b \Rightarrow a))$$

Define your own 2-valued semantics S_2 for $\mathcal{L}$, such that **none of** A_1, A_2 is a S_2 **tautology**

Verify your results. You **can use shorthand** notation.

Solution

This is not the only solution, but it is the simplest and most obvious I could think of! Here it is.

We define S_2 **connectives** as follows

$\neg x = F, \quad x \Rightarrow y = F, \quad x \cup y = F$ for all $x, y \in \{F, T\}$

Obviously, for any v,

$$v^*(a \cup \neg a) = F \text{ and } v^*(a \Rightarrow (b \Rightarrow a)) = F$$

Question 11

Prove using proper **classical** logical equivalences (list them at each step) that for any formulas A, B of language $\mathcal{L}_{\{\neg,\, \cup,\, \Rightarrow\}}$

$$\neg(A \Leftrightarrow B) \equiv ((A \cap \neg B) \cup (\neg A \cap B))$$

Solution

$\neg(A \Leftrightarrow B) \equiv^{def} \neg((A \Rightarrow B) \cap (B \Rightarrow A))$

$\equiv^{deMorgan} (\neg(A \Rightarrow B) \cup \neg(B \Rightarrow A))$

$\equiv^{negimpl} ((A \cap \neg B) \cup (B \cap \neg A)) \equiv^{commut} ((A \cap \neg B) \cup (\neg A \cap B))$

Question 12

Question 12

Prove using proper **classical** logical equivalences (list them at each step) that for any formulas A, B of language $\mathcal{L}_{\{\neg,\ \cup,\ \Rightarrow\}}$

$$((B \cap \neg C) \Rightarrow (\neg A \cup B)) \equiv ((B \Rightarrow C) \cup (A \Rightarrow B))$$

Solution

$((B \cap \neg C) \Rightarrow (\neg A \cup B))$

$\equiv^{impl}(\neg(B \cap \neg C) \cup (\neg A \cup B))$

$\equiv^{deMorgan}((\neg B \cup \neg\neg C) \cup (\neg A \cup B))$

$\equiv^{dneg}((\neg B \cup C) \cup (\neg A \cup B)) \equiv^{impl}((B \Rightarrow C) \cup (A \Rightarrow B))$

Question 13

We **define** Ł connectives for $\mathcal{L}_{\{\neg,\ \cup,\ \Rightarrow\}}$ as follows

Ł **Negation** $\neg$ is a **function:**

$$\neg : \ \{T, \perp, F\} \longrightarrow \{T, \perp, F\}$$

such that $\neg \perp = \perp, \quad \neg T = F, \quad \neg F = T$

Ł **Conjunction** $\cap$ is a **function:**

$$\cap : \ \{T, \perp, F\} \times \{T, \perp, F\} \longrightarrow \{T, \perp, F\}$$

such that $x \cap y = min\{x,\ y\}$ for all $x, y \in \{T, \perp, F\}$

Remember that we assumed: $F < \perp < T$

Question 13

Ł Implication $\Rightarrow$ is a **function:**

$$\Rightarrow: \ \{T, \perp, F\} \times \{T, \perp, F\} \longrightarrow \{T, \perp, F\}$$

such that

$$x \Rightarrow y = \begin{cases} \neg x \cup y & \text{if } x > y \\ T & \text{otherwise} \end{cases}$$

Given a formula $((a \cap b) \Rightarrow \neg b) \in \mathcal{F}$ of $\mathcal{L}_{\{\neg,\ \cup,\ \Rightarrow\}}$

Use the fact that $v : VAR \longrightarrow \{F, \perp, T\}$ is such that $v^*(((a \cap b) \Rightarrow \neg b)) = \perp$ under **Ł** semantics **to evaluate** all possible $v^*(((b \Rightarrow \neg a) \Rightarrow (a \Rightarrow \neg b)) \cup (a \Rightarrow b))$

You **can** use shorthand notation

Question 13 Solution

Solution

The formula $((a \cap b) \Rightarrow \neg b) = \perp$ in Ł connectives semantics in

two cases written is the shorthand notation as

C1 $(a \cap b) = \perp$ and $\neg b = F$

C2 $(a \cap b) = T$ and $\neg b = \perp$.

Consider case **C1**

$\neg b = F$, so $v(b) = T$, and hence $(a \cap T) = v(a) \cap T = \perp$

if and only if $v(a) = \perp$

It means that $v^*(((a \cap b) \Rightarrow \neg b)) = \perp$ for any v, is such that

$v(a) = \perp$ and $v(b) = T$

Question 13 Solution

We now **evaluate** (in shorthand notation)

$v^*(((b \Rightarrow \neg a) \Rightarrow (a \Rightarrow \neg b)) \cup (a \Rightarrow b))$

$= (((T \Rightarrow \neg \perp) \Rightarrow (\perp \Rightarrow \neg T)) \cup (\perp \Rightarrow T)) = ((\perp \Rightarrow \perp) \cup T) = T$

Consider now **Case C2**

$\neg b = \perp$, i.e. $b = \perp$, and hence $(a \cap \perp) = T$ what is **impossible,** hence v from the **Case C1** is the only one

Question 14

Question 14

Use the **Definability of Conjunction** in terms of disjunction and negation **Equivalence**

$$(A \cap B) \equiv \neg(\neg A \cup \neg B)$$

to transform a formula

$$A = \neg(\neg(\neg a \cap \neg b) \cap a)$$

of the language $\mathcal{L}_{\{\cap,\neg\}}$ into a logically equivalent formula B of the language $\mathcal{L}_{\{\cup,\neg\}}$

Question 14

Solution

$$\neg(\neg(\neg a \cap \neg b) \cap a) \equiv \neg\neg(\neg\neg(\neg a \cap \neg b) \cup \neg a)$$

$$\equiv ((\neg a \cap \neg b) \cup \neg a) \equiv (\neg(\neg\neg a \cup \neg\neg b) \cup \neg a)$$

$$\equiv \neg(a \cup b) \cup \neg a)$$

The formula B of $\mathcal{L}_{\{\cup,\neg\}}$ equivalent to A is

$$B = (\neg(a \cup b) \cup \neg a)$$

Equivalence of Languages Definition

Definition

Given two languages: $\mathcal{L}_1 = \mathcal{L}_{CON_1}$ and $\mathcal{L}_2 = \mathcal{L}_{CON_2}$, for $CON_1 \neq CON_2$

We say that they are **logically equivalent**, i.e.

$$\mathcal{L}_1 \equiv \mathcal{L}_2$$

if and only if the following conditions **C1, C2** hold.

C1: for any formula A of $\mathcal{L}_1$, there is a formula B of $\mathcal{L}_2$, such that $A \equiv B$

C2: for any formula C of $\mathcal{L}_2$, there is a formula D of $\mathcal{L}_1$, such that $C \equiv D$

Question 14

Prove the logical equivalence of the languages

$$\mathcal{L}_{\{\neg,\cup\}} \equiv \mathcal{L}_{\{\neg,\Rightarrow\}}$$

Solution

We need **two definability equivalences**:

implication in terms of disjunction and negation

$$(A \Rightarrow B) \equiv (\neg A \cup B)$$

and disjunction in terms of implication negation,

$$(A \cup B) \equiv (\neg A \Rightarrow B)$$

and the **Substitution Theorem**

Question 15
Prove the logical equivalence of the languages

$$\mathcal{L}_{\{\neg,\cap,\cup,\Rightarrow\}} \equiv \mathcal{L}_{\{\neg,\cap,\cup\}}$$

Solution
We need only the **definability of implication** in terms of disjunction and negation equivalence

$$(A \Rightarrow B) \equiv (\neg A \cup B)$$

as the **Substitution Theorem** for any formula A of $\mathcal{L}_{\{\neg,\cap,\cup,\Rightarrow\}}$ **there is** a formula B of $\mathcal{L}_{\{\neg,\cap,\cup\}}$ such that $A \equiv B$ and the condition **C1** holds

Observe that any formula A of language $\mathcal{L}_{\{\neg,\cap,\cup\}}$ is also a formula of the language $\mathcal{L}_{\{\neg,\cap,\cup,\Rightarrow\}}$ and of course $A \equiv A$ so the condition **C2** also holds

Question 16

Prove that

$$\mathcal{L}_{\{\neg,\cap\}} \equiv \mathcal{L}_{\{\neg,\Rightarrow\}}$$

Solution

The equivalence of languages holds due to the following two **definability of connectives equivalences**, respectively

$$(A \cap B) \equiv \neg(A \Rightarrow \neg B), \qquad (A \Rightarrow B) \equiv \neg(A \cap \neg B)$$

and **Substitution Theorem**

Question 17

Prove that in classical semantics

$$\mathcal{L}_{\{\neg,\Rightarrow\}} \equiv \mathcal{L}_{\{\neg,\Rightarrow,\cup\}}$$

Solution

OBSERVE that the condition **C1** holds because any formula of $\mathcal{L}_{\{\neg,\Rightarrow\}}$ is also a formula of $\mathcal{L}_{\{\neg,\Rightarrow,\cup\}}$

Condition **C2** holds due to the following definability of connectives equivalence

$$(A \cup B) \equiv (\neg A \Rightarrow B)$$

and **Substitution Theorem**

Question 18

Prove that the equivalence defining $\cup$ in terms of negation and implication in classical logic **does not hold** under **Ł** semantics, i.e. that

$$(A \cup B) \not\equiv_{\mathbf{L}} (\neg A \Rightarrow B)$$

but nevertheless

$$\mathcal{L}_{\{\neg,\Rightarrow\}} \equiv_{\mathbf{L}} \mathcal{L}_{\{\neg,\Rightarrow,\cup\}}$$

Question 18

Solution

We prove

$$\mathcal{L}_{\{\neg,\Rightarrow\}} \equiv_{\mathbf{L}} \mathcal{L}_{\{\neg,\Rightarrow,\cup\}}$$

as follows

Condition **C2** holds because the definability of connectives equivalence

$$(A \cup B) \equiv_{\mathbf{L}} ((A \Rightarrow B) \Rightarrow B)$$

Check it by verification as an exercise

C1 holds because any formula of $\mathcal{L}_{\{\neg,\Rightarrow\}}$ is a formula of $\mathcal{L}_{\{\neg,\Rightarrow,\cup\}}$

Observe that the equivalence $(A \cup B) \equiv (A \Rightarrow B) \Rightarrow B)$ provides also an alternative proof of **C2** in classical case