cse371/mat371
LOGIC

Professor Anita Wasilewska
LECTURE 3c
Chapter 3
Propositional Semantics: Classical and Many Valued

Extensional Semantics
Given a propositional language \( \mathcal{L}_\text{CON} \), the symbols for its connectives always have some intuitive meaning. A formal definition of the meaning of these symbols is called a semantics for the language \( \mathcal{L}_\text{CON} \).

A given language \( \mathcal{L}_\text{CON} \) can have different semantics but we always define them in order to single out special formulas of the language, called tautologies. Tautologies are formulas of the language that are always true under a given semantics.
We have already introduced the intuitive and formal notions of a classical **semantics**, discussed its **motivation** and underlying assumptions.

The **classical semantics** assumption is that it considers only **two** logical values. The other one is that all classical propositional **connectives** are **extensional**.

We have also observed that in everyday language there are expressions such as "I believe that", "it is possible that", "certainly", etc. and that they are represented by some propositional **connectives** which are **not extensional**.
Non-extensional connectives do not play any role in mathematics and so are not discussed in classical logic and will be studied separately.

The extensional connectives are defined intuitively as such that the logical value of the formulas form by means of these connectives and certain given formulas depends only on the logical value(s) of the given formulas.
We adopt a following **formal** definition of extensional connectives for a propositional language $\mathcal{L}_{CON}$

**Definition**

Let $\mathcal{L}_{CON}$ be such that $CON = C_1 \cup C_2$, where $C_1, C_2$ are the sets of **unary** and **binary** connectives, respectively. Let $LV$ be a non-empty set of **logical values**. A connective $\nabla \in C_1$ or $\circ \in C_2$ is called **extensional** if it is defined by a respective function

$$\nabla : LV \rightarrow LV \quad \text{or} \quad \circ : LV \times LV \rightarrow LV$$
A semantics $\mathcal{M}$ for a language $\mathcal{L}_{CON}$ is called extensional provided all connectives in $\text{CON}$ are extensional and its notion of tautology is defined terms of the connectives and their logical values.

A semantics with a set of $m$ logical values is called a $m$-valued extensional.

The classical semantics is a special case of a 2-valued extensional semantics.

Classical semantics defines classical logic with its set of classical propositional tautologies.

Many of logics are defined by various extensional semantics with sets of logical values $\text{LV}$ with more then 2 elements.
The languages of many important logics like modal, multi-modal, knowledge, believe, temporal, contain connectives that are not extensional because they are defined by non-extensional semantics.

The intuitionistic logic is based on the same language as the classical one and its Kripke Models semantics is not extensional.

Defining a semantics for a given language means more then defining connectives.

The ultimate goal of any semantics is to define the notion of its own tautology.
In order to define which formulas of a given $L_{CON}$ we want to to be tautologies under a given semantics $M$ we assume that the set $LV$ of logical values of $M$ always has a distinguished logical value, often denoted by $T$ for "absolute" truth.

We also can distinguish, and often we do, another special value $F$ representing "absolute" falsehood.

We will use these symbols $T$, $F$ for "absolute" truth and falsehood.

We may also use other symbols like $1$, $0$ or others.
The "absolute" truth value $T$ serves to define a notion of a tautology (as a formula always "true")

Extensional semantics share not only the similar pattern of defining their (extensional) connectives, but also the method of defining the notion of a tautology

We hence define a general notion of an extensional semantics as sequence of steps leading to the definition of a tautology
Here are the steps leading to the definition of a tautology

**Step 1** We define all extensional connectives of $M$

**Step 2** We define main component of the definition of a tautology, namely a function $v$ that assigns to any formula $A \in F$ its logical value from $LV$

The function $v$ is often called a truth assignment and we will use this name
Extensional Semantics $\mathbf{M}$ Introduction

Step 3  Given a truth assignment $v$ and a formula $A \in \mathcal{F}$, we define what does it mean that

$$v \text{ satisfies } A$$

i.e. we define a notion saying that $v$ is a model for $A$ under semantics $\mathbf{M}$

Step 4  We define a notion of tautology as follows

$A$ is a tautology under semantics $\mathbf{M}$ if and only if all truth assignments $v$ satisfy $A$

i.e. that all truth assignments $v$ are models for $A$
We use a notion of a model because it is an important, if not the most important notion of modern logic.

The notion of a model is usually defined in terms of the notion of satisfaction.

In classical propositional logic these notions are the same and the use of expressions
"v satisfies A" and "v is a model for A" is interchangeable.

This also is true for any propositional extensional semantics and in particular it holds for m-valued semantics discussed later in this chapter.
Extensional Semantics M Introduction

The notions of satisfaction and model are not interchangeable for predicate languages semantics.

We already discussed intuitively the notion of model and satisfaction for predicate language in chapter 2.

We will define them in full formality in chapter 8.

The use of the notion of a model also allows us to adopt and discuss the standard predicate logic definitions of consistency and independence for propositional case.
Definition

Any formal definition of an extensional semantics $M$ for a given language $L_{CON}$ consists of specifying the following steps defining its main components:

Step 1 We define a set $LV$ of logical values, its distinguished value $T$, and define all connectives of $L_{CON}$ to be extensional.

Step 2 We define notion of a truth assignment and its extension.

Step 3 We define notions of satisfaction, model, counter model.

Step 4 We define notion of a tautology under the semantics $M$. 

Extensional Semantics $M$ Formal Definition
Extensional Semantics $\mathbf{M}$ Formal Definition

What differs one semantics from the other is the choice of the set $LV$ of logical values and definition of the connectives of $\mathcal{L}_{CON}$, that are defined in the first step below.

**Step 1** We adopt a following formal definition of extensional connectives of $\mathcal{L}_{CON}$

**Definition**
Let $\mathcal{L}_{CON}$ be such that $\text{CON} = C_1 \cup C_2$, where $C_1, C_2$ are the sets of unary and binary connectives, respectively.
Let $LV$ be a non-empty set of logical values.
A connective $\nabla \in C_1$ or $\circ \in C_2$ is called extensional if it is defined by a respective function

$$\nabla : LV \rightarrow LV \quad \text{or} \quad \circ : LV \times LV \rightarrow LV$$
**M Truth Assignment Formal Definition**

**Step 2** We define a function called **truth assignment** and its **extension** in terms of the **propositional connectives** as defined in the **Step 1**

**Definition**
Let \( LV \) be the set of logical values of \( M \) and \( VAR \) the set of propositional variables of the language \( L_{CON} \)

Any function

\[
v : VAR \rightarrow LV
\]

is called a **truth assignment** under semantics \( M \)

We call it for short a **M truth assignment**

We use the term **M truth assignment** and **M truth extension** to stress that it is defined **relatively** to a given semantics \( M \)
Definition
Given a \( M \) truth assignment \( v : \text{VAR} \rightarrow \text{LV} \)
We define its **extension** \( v^* \) to the set \( \mathcal{F} \) of all formulas of \( \mathcal{L}_{\text{CON}} \) as any function

\[
v^* : \mathcal{F} \rightarrow \text{LV}
\]

such that the following conditions are satisfied.

(i) for any \( a \in \text{VAR} \),

\[
v^*(a) = v(a);
\]

(ii) For any connectives \( \nabla \in C_1, \circ \in C_2 \), and for any formulas \( A, B \in \mathcal{F} \),

\[
v^*(\nabla A) = \nabla v^*(A) \quad \text{and} \quad v^*((A \circ B)) = \circ(v^*(A), v^*(B))
\]

We call the \( v^* \) the **\( M \) truth extension**
M Truth Extension Formal Definition

Remark

The symbols on the left-hand side of the equations

\[ v^*(\nabla A) = \nabla v^*(A) \] and \[ v^*((A \circ B)) = \circ(v^*(A), v^*(B)) \]

represent connectives in their natural language meaning and the symbols on the right-hand side represent connectives in their semantical meaning as defined in the Step1.
We use names "\textit{M truth assignment}" and "\textit{M truth extension}" to stress that we define them for the set of logical values of the semantics \textbf{M}.

\textbf{Notation Remark}

For any function \( g \), we use a symbol \( g^* \) to denote its extension to a larger domain.

Mathematician often use the same symbol \( g \) for both a function \( g \) and its extension \( g^* \).
Step 3  The notions of satisfaction and model are interchangeable in M semantics and we define them as follows.

Definition

Given an M truth assignment \( v : \text{VAR} \rightarrow \text{LV} \) and its M truth extension \( v^* \). Let \( T \in \text{LV} \) be the distinguished logical truth value.

We say that the truth assignment \( v \) M satisfies a formula \( A \) if and only if \( v^*(A) = T \).

We write symbolically

\[ v \models_{\text{M}} A \]

Any truth assignment \( v \), such that \( v \models_{\text{M}} A \) is called an M model for the formula \( A \).
Counter Model

Definition
Given an $\mathbf{M}$ truth assignment $\nu : \text{VAR} \rightarrow \text{LV}$ and its $\mathbf{M}$ truth extension $\nu^*$. Let $T \in \text{LV}$ be the distinguished logical truth value

We say that the truth assignment $\nu$ $\mathbf{M}$ does not satisfy a formula $A$ if and only if $\nu^*(A) \neq T$

We write symbolically

$$\nu \not\models_{\mathbf{M}} A$$

Any truth assignment $\nu$, such that $\nu \not\models_{\mathbf{M}} A$ is called an $\mathbf{M}$ counter model for the formula $A$
**M Tautology**

**Step 4** We define the notion of **M tautology** as follows

**Definition**
A formula $A$ is an **M tautology** if and only if $\forall v: \mathsf{VAR} \rightarrow \mathsf{LV}, v \models_M A$, for all truth assignments $v : \mathsf{VAR} \rightarrow \mathsf{LV}$

We denote it as $\models_M A$

We also say that $A$ is an **M tautology** if and only if all truth assignments $v : \mathsf{VAR} \rightarrow \mathsf{LV}$ are **M models** for $A$
Observe that directly from definition of the **M model** we get the following equivalent form of the definition of **tautology**

**Definition**

A formula $A$ is an **M tautology** if and only if $v^*(A) = T$, for all truth assignments $v : VAR \rightarrow LV$

We denote by **MT** the set of **all tautologies** under the semantic **M**, i.e.

$$MT = \{ A \in F : \vdash_M A \}$$
M Tautology

Obviously, when we develop a logic by defining its semantics we want the semantics to be such that the logic has a non empty set of its tautologies. We express it in a form of the following definition.

**Definition**

Given a language $\mathcal{L}_{CON}$ and its extensional semantics $M$

The semantics $M$ is **well defined** if and only if its set $MT$ of all tautologies is non empty, i.e. when

$$MT \neq \emptyset$$
Extensional Semantics $M$

As the next steps we use the definitions established here to define and discuss in details the following particular cases of the extensional semantics $M$.

Many valued semantics have their beginning in the work of Łukasiewicz (1920).

He was the first to define a 3-valued extensional semantics for a language $\mathcal{L}_{\{\neg, \cap, \cup, \Rightarrow\}}$ of classical logic, and called it a 3-valued logic, for short.
Extensional Semantics

The other logics defined by various extensional semantics followed and we will discuss some of them.

In particular we present Heyting’s 3-valued semantics as an introduction to the discussion of first ever semantics for the intuitionistic logic and some modal logics.
Challenge Exercise

1. Define your own propositional language $L_{CON}$ that contains also different connectives that the standard connectives $\neg, \cup, \cap, \Rightarrow$

Your language $L_{CON}$ does not need to include all (if any!) of the standard connectives $\neg, \cup, \cap, \Rightarrow$

2. Describe intuitive meaning of the new connectives of your language

3. Give some motivation for your own semantic $M$

4. Define formally your own extensional semantics $M$ for your language $L_{CON}$

Write carefully all Steps 1-4 of the definition of your $M$