cse371/mat371
LOGIC

Professor Anita Wasilewska
LECTURE 3a
Chapter 3
Propositional Semantics: Classical and Many Valued

Classical Semantics
Semantics- General Principles

Given a propositional language $\mathcal{L} = \mathcal{L}_{CON}$

Symbols for connectives of $\mathcal{L}$ always have some intuitive meaning

**Semantics** provides a formal definition of the meaning of these symbols

It provides a method of defining formally a notion of tautology under a given **semantics**
Extensional Connectives

In Chapter 2 we described the intuitive classical propositional semantics and introduced the following notion of extensional connectives:

**Extensional connectives** are the propositional connectives that have the following property: the logical value of the formulas formed by means of these connectives and certain given formulas depends only on the logical value(s) of the given formulas.

We also assumed that all classical propositional connectives

\[ \neg, \lor, \land, \Rightarrow, \equiv, \uparrow, \downarrow \]

are extensional.
Non-Extensional Connectives

We have also observed the following

**Remark**
In everyday language there are expressions such as
"I believe that", "it is possible that", "certainly", etc.
They are represented by some propositional connectives
which are not extensional

**Non-extensional** connectives do not play any role in
**mathematics** and so are not discussed in **classical logic**
and will be studied separately
Definition of Extensional Connectives

Given a propositional language $L_{CON}$ for the set $CON = C_1 \cup C_2$, where $C_1$ is the set of all unary connectives, and $C_2$ is the set of all binary connectives. Let $V$ be a non-empty set of logical values.

We adopt now a following formal definition of extensional connectives.

**Definition**

Connectives $\triangle \in C_1$, $\circ \in C_2$ are called extensional if and only if their semantics is defined by respective functions:

$$\triangle : V \rightarrow V \quad \text{and} \quad \circ : V \times V \rightarrow V$$
Functional Dependency and Definability of Connectives

In Chapter 2 we talked about functional dependency of connectives and of definability of a connective in terms of other connectives.

We define these notions formally as follows.
Functional Dependency and Definability of Connectives

Given a propositional language $\mathcal{L}_{CON}$ and an extensional semantics for it; i.e a semantics such that all connectives in $\mathcal{L}$ are extensional

**Definition**

Connectives $\circ \in CON$ and $\circ_1, \circ_2, \ldots, \circ_n \in CON$ (for $n \geq 1$) are functionally dependent iff $\circ$ is a certain function composition of functions $\circ_1, \circ_2, \ldots, \circ_n$

**Definition**

A connective $\circ \in CON$ is definable in terms of some connectives $\circ_1, \circ_2, \ldots, \circ_n \in CON$ iff $\circ \in CON$ and $\circ_1, \circ_2, \ldots, \circ_n \in CON$ are functionally dependent
Classical Propositional Semantics Assumptions

Assumptions

A1: We define our semantics for the language

\[ L = L\{\neg, \cup, \cap, \Rightarrow, \Leftrightarrow\} \]

A2: Two values: the set of logical values \( V = \{T, F\} \)

Logical values \( T, F \) denote truth and falsehood, respectively.

There are other notations, for example \( 0, 1 \)

A3: Extensionality: all connectives of \( L \) are extensional

Semantics for any language \( L \) for which the assumption A3 holds is called extensional semantics
Propositional Semantics Definition

Formal definition of a propositional extensional semantics for a given language $L_{CON}$ consists of providing definitions of the following four main components:

1. Extensional Connectives
2. Truth Assignment
3. Satisfaction, Model, Counter-Model
4. Tautology

The definition of the classical semantics and extensional semantics for some non-classical logics considered here will follow the same pattern.
Semantics: Classical Connectives Definition

Semantics Definition **Step 1**
The assumption of **extensionality of connectives** means that **unary connectives** are functions defined on a set \{T, F\} with values in the set \{T, F\} and

**binary connectives** are functions defined on a set \{T, F\} \times \{T, F\} with values in the set \{T, F\}

In particular we adopt the following definitions

**Negation Definition**

**Negation** \( \neg \) **is a function:**

\[
\neg : \{T, F\} \longrightarrow \{T, F\},
\]

such that

\[
\neg T = F, \quad \neg F = T
\]
Semantics: Classical Connectives Definition

Notation
When defining connectives as functions we usually write the name of a function (our connective) **between the arguments**, not in front as in function notation, i.e. for example we write $T \cap T = T$ instead of $\cap(T, T) = T$

Conjunction Definition
**Conjunction** $\cap$ is a **function**:

$$\cap : \{T, F\} \times \{T, F\} \rightarrow \{T, F\},$$

such that

$$\cap(T, T) = T, \quad \cap(T, F) = F, \quad \cap(F, T) = F, \quad \cap(F, F) = F$$

We write it as

$$T \cap T = T, \quad T \cap F = F, \quad F \cap T = F, \quad F \cap F = F$$
Semantics: Classical Connectives Definition

Disjunction Definition

**Disjunction** \( \cup \) is a **function**:

\[
\cup : \{T, F\} \times \{T, F\} \longrightarrow \{T, F\}
\]

such that

\[
\begin{align*}
\cup(T, T) &= T, & \cup(T, F) &= T, & \cup(F, T) &= T, & \cup(F, F) &= F
\end{align*}
\]

We write it as

\[
T \cup T = T, \quad T \cup F = T, \quad F \cup T = T, \quad F \cup F = F
\]
Semantics: Classical Connectives Definition

Implication Definition

Implication $\Rightarrow$ is a function:

$$\Rightarrow: \{T, F\} \times \{T, F\} \rightarrow \{T, F\}$$

such that

$$\Rightarrow (T, T) = T, \quad \Rightarrow (T, F) = F, \quad \Rightarrow (F, T) = T, \quad \Rightarrow (F, F) = T$$

We write it as

$$T \Rightarrow T = T, \quad T \Rightarrow F = F, \quad F \Rightarrow T = T, \quad F \Rightarrow F = T$$
Equivalence Definition


such that

\[ \Leftrightarrow (T, T) = T, \ \Leftrightarrow (T, F) = F, \ \Leftrightarrow (F, T) = F, \ \Leftrightarrow (T, T) = T \]

We write it as

\[ T \Leftrightarrow T = T, \ \ T \Leftrightarrow F = F, \ \ F \Leftrightarrow T = F, \ \ T \Leftrightarrow T = T \]
Classical Connectives Truth Tables

We write the functions defining connectives in a form of tables, usually called the classical truth tables

Negation:

\( \neg T = F, \quad \neg F = T \)

\[ \begin{array}{c|cc}
\neg & T & F \\
\hline
T & F & T \\
\end{array} \]

Conjunction:

\( T \cap T = T, \quad T \cap F = F, \quad F \cap T = F, \quad F \cap F = F \)

\[ \begin{array}{c|ccc}
\cap & T & F \\
\hline
T & T & F \\
F & F & F \\
\end{array} \]
Classical Connectives Truth Tables

Disjunction:

\[ T \cup T = T, \quad T \cup F = T, \quad F \cup T = T, \quad F \cup F = F \]

<table>
<thead>
<tr>
<th>( \cup )</th>
<th>T</th>
<th>F</th>
</tr>
</thead>
<tbody>
<tr>
<td>T</td>
<td>T</td>
<td>T</td>
</tr>
<tr>
<td>F</td>
<td>T</td>
<td>F</td>
</tr>
</tbody>
</table>

Implication:

\[ T \Rightarrow T = T, \quad T \Rightarrow F = F, \quad F \Rightarrow T = T, \quad F \Rightarrow F = T \]

<table>
<thead>
<tr>
<th>( \Rightarrow )</th>
<th>T</th>
<th>F</th>
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</thead>
<tbody>
<tr>
<td>T</td>
<td>T</td>
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<tr>
<td>F</td>
<td>T</td>
<td>T</td>
</tr>
</tbody>
</table>
Classical Connectives Truth Tables

Equivalence:

\[ T \iff T = T, \ T \iff F = F, \ F \iff T = F, \ F \iff F = T \]

<table>
<thead>
<tr>
<th>( \iff )</th>
<th>T</th>
<th>F</th>
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<tbody>
<tr>
<td>T</td>
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<td>F</td>
</tr>
<tr>
<td>F</td>
<td>F</td>
<td>T</td>
</tr>
</tbody>
</table>

This ends the **Step 1** of the semantics definition
Definability of Classical Connectives

We adopted the following definition

Definition

A connective $\circ \in CON$ is **definable** in terms of some connectives $\circ_1, \circ_2, ... \circ_n \in CON$ iff $\circ$ is a certain function composition of functions $\circ_1, \circ_2, ... \circ_n$

Example

Classical implication $\Rightarrow$ is **definable** in terms of $\cup$ and $\neg$ because $\Rightarrow$ can be defined as a **composition** of functions $\neg$ and $\cup$

More precisely, a function $h : \{T, F\} \times \{T, F\} \rightarrow \{T, F\}$ defined by a formula

$$h(x, y) = \cup(\neg x, y)$$

is a **composition of functions** $\neg$ and $\cup$ and we prove that the implication function $\Rightarrow$ is equal with $h$
Short Review: Equality of Functions

Definition
Given two sets $A, B$ and functions $f, g$ such that

$$f : A \rightarrow B \quad \text{and} \quad g : A \rightarrow B$$

We say that the functions $f, g$ are equal and write is as $f = g$ if and only if $f(x) = g(x)$ for all elements $x \in A$

Example: Consider functions

$$\Rightarrow: \{T, F\} \times \{T, F\} \rightarrow \{T, F\} \quad \text{and} \quad h : \{T, F\} \times \{T, F\} \rightarrow \{T, F\}$$

where $\Rightarrow$ is classical implication and the function $h$ is defined by the formula $h(x, y) = \cup(\neg x, y)$

We prove that $\Rightarrow = h$ by evaluating that

$$\Rightarrow (x, y) = h(x, y) = \cup(\neg x, y), \quad \text{for all} \ (x, y) \in \{T, F\} \times \{T, F\}$$
Definability of Classical Implication

We re-write formula \( \Rightarrow (x, y) = \cup(\neg x, y) \) in our adopted notation as

\[ x \Rightarrow y = \neg x \cup y \quad \text{for all} \quad (x, y) \in \{T, F\} \times \{T, F\} \]

and call it a formula defining \( \Rightarrow \) in terms of \( \cup \) and \( \neg \).

We verify correctness of the definition as follows

- \( T \Rightarrow T = T \) and \( \neg T \cup T = F \cup T = T \) \hspace{1cm} yes
- \( T \Rightarrow F = F \) and \( \neg T \cup F = F \cup F = F \) \hspace{1cm} yes
- \( F \Rightarrow F = T \) and \( \neg F \cup F = T \cup F = T \) \hspace{1cm} yes
- \( F \Rightarrow T = T \) and \( \neg F \cup T = T \cup T = T \) \hspace{1cm} yes
Definability of Classical Connectives

Exercise 1
Find a formula defining $\land, \iff$ in terms of $\lor$ and $\neg$

Exercise 2
Find a formula defining $\Rightarrow, \lor, \iff$ in terms of $\land$ and $\neg$

Exercise 3
Find a formula defining $\land, \lor, \iff$ in terms of $\Rightarrow$ and $\neg$

Exercise 4
Find a formula defining \lor in terms of $\Rightarrow$ alone
Two More Classical Connectives

Sheffer Alternative Negation \(\uparrow\)

\[
\uparrow: \quad \{T, F\} \times \{T, F\} \rightarrow \{T, F\}
\]

such that

\[
T \uparrow T = F, \quad T \uparrow F = T, \quad F \uparrow T = T, \quad F \uparrow F = T
\]

Łukasiewicz Joint Negation \(\downarrow\)

\[
\downarrow: \quad \{T, F\} \times \{T, F\} \rightarrow \{T, F\}
\]

such that

\[
T \downarrow T = F, \quad T \downarrow F = F, \quad F \downarrow T = F, \quad F \downarrow F = T
\]
Definability of Classical Connectives

Exercise 4
Show that the **Sheffer Alternative Negation** \( \uparrow \) defines all classical connectives \( \neg, \Rightarrow, \lor, \land, \iff \)

Exercise 5
Show that **Łukasiewicz Joint Negation** \( \downarrow \) defines all classical connectives \( \neg, \Rightarrow, \lor, \land, \iff \)

Exercise 6
Show that the two binary connectives: \( \downarrow \) and \( \uparrow \) suffice, each of them separately, to define **all classical connectives**, whether unary or binary
Step 2
We define the next components of the classical propositional semantics in terms of the propositional connectives as defined in the Step 1 and a function called truth assignment.

Definition
A truth assignment is any function

\[ \nu : \text{VAR} \rightarrow \{T, F\} \]

Observe that the domain of truth assignment is the set of propositional variables, i.e. the truth assignment is defined only for atomic formulas.
Truth Assignment Extension

We now extend the truth assignment $v$ to the set of all formulas $\mathcal{F}$ in order define formally the logical value for any formula $A \in \mathcal{F}$.

The definition of the extension of the variable assignment $v$ to the set $\mathcal{F}$ follows the same pattern for the all extensional connectives, i.e. for all extensional semantics.
Truth Assignment Extension $v^*$ to $\mathcal{F}$

Definition
Given the truth assignment

$$v : \text{VAR} \rightarrow \{T, F\}$$

We define its \textbf{extension} $v^*$ to the set $\mathcal{F}$ of all formulas of $\mathcal{L}$ as any function

$$v^* : \mathcal{F} \rightarrow \{T, F\}$$

such that the following conditions are satisfied

(i) for any $a \in \text{VAR}$ (atomic formula)

$$v^*(a) = v(a);$$
Truth Assignment Extension $v^*$ to $\mathcal{F}$

(ii) and for any $A, B \in \mathcal{F}$ we put

$$v^*(\neg A) = \neg v^*(A);$$

$$v^*((A \cap B)) = \cap(v^*(A), v^*(B));$$

$$v^*((A \cup B)) = \cup(v^*(A), v^*(B));$$

$$v^*((A \Rightarrow B)) = \Rightarrow(v^*(A), v^*(B));$$

$$v^*((A \Leftrightarrow B)) = \Leftrightarrow(v^*(A), v^*(B))$$

The symbols on the **left-hand side** of the equations represent connectives in their **natural language meaning** and the symbols on the **right-hand side** represent connectives in their **semantical meaning** given by the classical truth tables.
Extension $v^*$ Definition Revisited

**Notation**
For binary connectives (two argument functions) we adopt a convention to write the symbol of the connective (name of the 2 argument function) between its arguments as we do in a case arithmetic operations.

The condition (ii) of the definition of the extension $v^*$ can be hence written as follows:

(ii) and for any $A, B \in \mathcal{F}$ we put

$$v^*(\neg A) = \neg v^*(A);$$
$$v^*((A \cap B)) = v^*(A) \cap v^*(B);$$
$$v^*((A \cup B)) = v^*(A) \cup v^*(B);$$
$$v^*((A \Rightarrow B)) = v^*(A) \Rightarrow v^*(B);$$
$$v^*((A \Leftrightarrow B)) = v^*(A) \Leftrightarrow v^*(B)$$

We will use this notation for the rest of the book.
Truth Assignment Extension Example

Consider a formula

\[ ((a \Rightarrow b) \cup \neg a) \]

and a truth assignment \( v \) such that

\[ v(a) = T, \quad v(b) = F \]

Observe that we did not specify \( v(x) \) of any \( x \in VAR - \{a, b\} \), as these values do not influence the computation of the logical value \( v^*(A) \) of the formula \( A \)

We say: ”\( v \) such that” - as we consider its values for the set \( \{a, b\} \subseteq VAR \)

Nevertheless, the domain of \( v \) is the set of all variables \( VAR \) and we have to remember that.
Truth Assignment Extension Example

Given a formula \( A: ((a \Rightarrow b) \cup \neg a)) \) and a truth assignment \( v \) such that \( v(a) = T, \ v(b) = F \)

We calculate the logical value of the formula \( A \) as follows:

\[
\begin{align*}
\nu^*(A) &= \nu^*((((a \Rightarrow b) \cup \neg a))) = \bigcup (\nu^*(a \Rightarrow b), \nu^*(\neg a)) = \\
&= \bigcup (\Rightarrow (\nu^*(a), \nu^*(b)), \neg \nu^*(a))) = \bigcup (\Rightarrow (v(a), v(b)), \neg v(a))) = \\
&= \bigcup (\Rightarrow (T, F), \neg T)) = \bigcup (F, F) = F
\end{align*}
\]

We can also calculate it as follows:

\[
\begin{align*}
\nu^*(A) &= \nu^*((((a \Rightarrow b) \cup \neg a))) = \nu^*(a \Rightarrow b) \cup \nu^*(\neg a) = \\
&= (v(a) \Rightarrow v(b)) \cup \neg v(a) = (T \Rightarrow F) \cup \neg T = F \cup F = F
\end{align*}
\]

We write it in a short-hand notation as

\( (T \Rightarrow F) \cup \neg T = F \cup F = F \)

On tests I will specify when you can use the the short-hand notation.
Semantics: Satisfaction Relation

Step 3

Definition: Let \( v : VAR \longrightarrow \{T, F\} \)

We say that
\( v \) satisfies a formula \( A \in \mathcal{F} \) iff \( v^*(A) = T \)

Notation: \( v \models A \)

Definition: We say that
\( v \) does not satisfy a formula \( A \in \mathcal{F} \) iff \( v^*(A) \neq T \)

Notation: \( v \not\models A \)

The relation \( \models \) is called a satisfaction relation
Semantics: Satisfaction Relation

Observe that $v^*(A) \neq T$ is equivalent to the fact that $v^*(A) = F$ only in 2-valued semantics and

$$v \not\models A \iff v^*(A) = F$$

Definition

We say that \textbf{v falsifies} the formula $A$ iff $v^*(A) = F$

Remark

For any formula $A \in \mathcal{F}$

$v \not\models A \iff v \textbf{ falsifies} the formula A$
Examples

Example 1: Let $A = ((a \Rightarrow b) \cup \neg a))$ and $v : VAR \longrightarrow \{T, F\}$ be such that $v(a) = T, v(b) = F$. We calculate $v^*(A)$ using a short hand notation as follows:

$$(T \Rightarrow F) \cup \neg T = F \cup F = F$$

By definition

$$v \not\models ((a \Rightarrow b) \cup \neg a))$$

Observe that we did not need to specify the $v(x)$ of any $x \in VAR - \{a, b\}$, as these values do not influence the computation of the logical value $v^*(A)$.
Examples

Example 2  Let $A = ((a \cap \neg b) \cup \neg c)$ and $v : VAR \longrightarrow \{T, F\}$ be such that $v(a) = T$, $v(b) = F$, $v(c) = T$

We calculate $v^*(A)$ using a short hand notation as follows

$$(T \cap \neg F) \cup \neg T = (T \cap T) \cup F = T \cup F = T$$

By definition

$v \models ((a \cap \neg b) \cup \neg c)$
Examples

Example 3  Let \( A = ((a \cap \neg b) \cup \neg c) \)
Consider now \( \nu_1 : \text{VAR} \rightarrow \{T, F\} \) such that
\( \nu_1(a) = T, \nu_1(b) = F, \nu_1(c) = T \) and
\( \nu_1(x) = F, \) for all \( x \in \text{VAR} - \{a, b, c\} \)
Observe that
\( \nu(a) = \nu_1(a), \nu(b) = \nu_1(b), \nu(c) = \nu_1(c) \)
Hence we get
\( \nu_1 \models ((a \cap \neg b) \cup \neg c) \)
Examples

Example 4  Let $A = ((a \cap \neg b) \cup \neg c)$
Consider now $v_2 : \text{VAR} \rightarrow \{T, F\}$ such that $v_2(a) = T, v_2(b) = F, v_2(c) = T, v_2(d) = T$ and $v_1(x) = F$, for all $x \in \text{VAR} - \{a, b, c, d\}$
Observe that $v(a) = v_2(a), v(b) = v_2(b), v(c) = v_2(c)$
Hence we get
$v_2 \models ((a \cap \neg b) \cup \neg c)$
Semantics: Model, Counter-Model

Definition:
Given a formula $A \in \mathcal{F}$ and $v : \text{VAR} \rightarrow \{T, F\}$

Any $v$ such that $v \models A$ is called a model for $A$

Any $v$ such that $v \not\models A$ is called a counter model for $A$

Observe that all truth assignments $v, v_1, v_2$ from our Examples 2, 3, 4 are models for the same formula $A$
Semantics: Tautology

Step 4

Definition:
For any formula \( A \in \mathcal{F} \)
\( A \) is a tautology iff \( v^*(A) = T \), for all \( v : VAR \rightarrow \{ T, F \} \)

i.e. we have that
\( A \) is a tautology iff any \( v : VAR \rightarrow \{ T, F \} \) is a model for \( A \)

Notation
We write symbolically \( \models A \) for the statement ”\( A \) is a tautology”
Semantics: not a tautology

Definition

A is not a tautology iff there is v, such that $v^*(A) \neq T$

i.e. we have that

A is not a tautology iff A has a counter-model

Notation

We write $\not\models A$ to denote the statement “A is not a tautology”
How Many

We just saw from the Examples 2, 3, 4 that given a model $v$ for a formula $A$, we defined 2 other models for $A$

These models were identical with $v$ on the variables in the formula $A$

Visibly we can keep constructing in a similar way more and more of such models

A natural question arises:

Given a model for a the formula $A$, how many other models for $A$ can be constructed?

The same question can be asked about counter-models for $A$, if they exist
**Challenge Problem**

**Challenge Problem :** prove the following

**Model Theorem**

For any formula $A \in \mathcal{F}$,

If $A$ has a **model** (counter-model), then it has uncountably many (exactly as many as real numbers) of **models** (counter-models)
Here is a more general question

**Question**

Given a formula $A \in \mathcal{F}$, how many truth assignments we have to consider to prove that the formula $A$ is a **tautology**?

**We prove** that there are as many of such truth assignments as real numbers.

But FORTUNATELY only a finite number of them is differs on the variables included in the formula $A$ and we do have the following

**Tautology Decidability Theorem**

The notion of classical propositional tautology $\models A$ is **decidable**
Restricted Truth Assignments

To address and to answer these questions formally we first introduce some notations and definitions.

**Notation:** for any formula $A$, we denote by $VAR_A$ a set of all variables that appear in $A$.

**Definition:** Given $v : VAR \rightarrow \{T, F\}$, any function $v_A : VAR_A \rightarrow \{T, F\}$ such that $v(a) = v_A(a)$ for all $a \in VAR_A$ is called a restriction of $v$ to the formula $A$.

**Fact 1**

For any formula $A$, any $v$, and its restriction $v_A$

\[ v \models A \text{ iff } v_A \models A \]
**Restricted Model**

**Definition:** Given a formula $A \in \mathcal{F}$, any function

$$w : \text{VAR}_A \rightarrow \{T, F\}$$

is called a truth assignment **restricted** to $A$

**Definition**

Given a formula $A \in \mathcal{F}$

Any function

$$w : \text{VAR}_A \rightarrow \{T, F\} \quad \text{such that} \quad w^*(A) = T$$

is called a **restricted MODEL** for $A$
Example

Example

\[ A = ((a \cap \neg b) \cup \neg c) \]

\[ VAR_A = \{a, b, c\} \]

Truth assignment **restricted** to \( A \) is any function:

\[ w : \{a, b, c\} \rightarrow \{T, F\}. \]

We use the following theorem to count all possible truth assignment **restricted** to \( A \)
Counting Functions

Counting Functions Theorem

For any finite sets A and B, if the set A has n elements and B has m elements, then there are \( m^n \) possible functions that map A into B.

Proof by Mathematical Induction over m

Example:
There are \( 2^3 = 8 \) truth assignments \( w \) restricted to

\[ A = ((a \Rightarrow \neg b) \cup \neg c) \]
Counting Theorem

For any $A \in \mathcal{F}$, there are $2^{|\text{VAR}_A|}$ possible truth assignments restricted to $A$. 
Example

Let \( A = ((a \cap \neg b) \cup \neg c) \)

All \( w \) restricted to \( A \) are listed in the table below

<table>
<thead>
<tr>
<th>( w )</th>
<th>( a )</th>
<th>( b )</th>
<th>( c )</th>
<th>( w^*(A) ) computation</th>
<th>( w^*(A) )</th>
</tr>
</thead>
<tbody>
<tr>
<td>( w_1 )</td>
<td>T</td>
<td>T</td>
<td>T</td>
<td>((T \Rightarrow T) \cup \neg T = T \cup F = T)</td>
<td>T</td>
</tr>
<tr>
<td>( w_2 )</td>
<td>T</td>
<td>T</td>
<td>F</td>
<td>((T \Rightarrow T) \cup \neg F = T \cup T = T)</td>
<td>T</td>
</tr>
<tr>
<td>( w_3 )</td>
<td>T</td>
<td>F</td>
<td>F</td>
<td>((T \Rightarrow F) \cup \neg F = F \cup T = T)</td>
<td>T</td>
</tr>
<tr>
<td>( w_4 )</td>
<td>F</td>
<td>F</td>
<td>T</td>
<td>((F \Rightarrow F) \cup \neg T = T \cup F = T)</td>
<td>T</td>
</tr>
<tr>
<td>( w_5 )</td>
<td>F</td>
<td>T</td>
<td>T</td>
<td>((F \Rightarrow T) \cup \neg T = T \cup F = T)</td>
<td>T</td>
</tr>
<tr>
<td>( w_6 )</td>
<td>F</td>
<td>T</td>
<td>F</td>
<td>((F \Rightarrow T) \cup \neg F = T \cup T = T)</td>
<td>T</td>
</tr>
<tr>
<td>( w_7 )</td>
<td>T</td>
<td>F</td>
<td>T</td>
<td>((T \Rightarrow F) \cup \neg T = F \cup F = F)</td>
<td>F</td>
</tr>
<tr>
<td>( w_8 )</td>
<td>F</td>
<td>F</td>
<td>F</td>
<td>((F \Rightarrow F) \cup \neg F = T \cup T = T)</td>
<td>T</td>
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</tbody>
</table>

\( w_1, w_2, w_3, w_4, w_5, w_6, w_8 \) are restricted models for \( A \)

\( w_7 \) is a restricted counter-model for \( A \)
Restrictions and Extensions

Given a formula $A$ and $w : \text{VAR}_A \rightarrow \{T, F\}$

**Definition**
Any function $v$, such that $v : \text{VAR} \rightarrow \{T, F\}$ and $v(a) = w(a)$, for all $a \in \text{VAR}_A$ is called an **extension** of $w$ to the set $\text{VAR}$ of all propositional variables

**Fact 2**
For any formula $A$, any $w$ **restricted** to $A$, and any of its extensions $v$

$$w \models A \iff v \models A$$
Tautology and Decidability

By the definition of a tautology and Facts 1, 2 we get the following

**Tautology Theorem**

\[
\models A \iff w \models A \quad \text{for all} \quad w : \text{VAR}_A \rightarrow \{T, F\}
\]

From above and the **Counting Theorem** we get

**Tautology Decidability Theorem**

The notion of classical propositional tautology \( \models A \) is **decidable**
We just PROVED correctness of the well known Truth Table Tautology Verification Method:
to verify whether $\models A$ list and evaluate all possible truth assignments $w$ restricted to $A$ and we have that
$\models A$ if all $w$ evaluate to $T$
$\not\models A$ if there is one $w$ that evaluates to $F$
Truth Table Example

Consider a formula $A$:

$$(a \Rightarrow (a \cup b))$$

We write the Truth Table:

<table>
<thead>
<tr>
<th>$w$</th>
<th>$a$</th>
<th>$b$</th>
<th>$w^*(A)$ computation</th>
<th>$w^*(A)$</th>
</tr>
</thead>
<tbody>
<tr>
<td>$w_1$</td>
<td>T</td>
<td>T</td>
<td>$(T \Rightarrow (T \cup T)) = (T \Rightarrow T) = T$</td>
<td>T</td>
</tr>
<tr>
<td>$w_2$</td>
<td>T</td>
<td>F</td>
<td>$(T \Rightarrow (T \cup F)) = (T \Rightarrow T) = T$</td>
<td>T</td>
</tr>
<tr>
<td>$w_3$</td>
<td>F</td>
<td>T</td>
<td>$(F \Rightarrow (F \cup T)) = (F \Rightarrow T) = T$</td>
<td>T</td>
</tr>
<tr>
<td>$w_4$</td>
<td>F</td>
<td>F</td>
<td>$(F \Rightarrow (F \cup F)) = (F \Rightarrow F) = T$</td>
<td>T</td>
</tr>
</tbody>
</table>

We evaluated that for all $w$ restricted to $A$, i.e. all functions $w : VAR_A \rightarrow \{T, F\}$, $w \models A$

This proves by **Tautology Theorem**

$$\models (a \Rightarrow (a \cup b))$$
Imagine now that $A$ has for example 200 variables. To find whether $A$ is a tautology by using the Truth Table Method one would have to evaluate 200 variables long expressions - not to mention that one would have to list $2^{200}$ restricted truth assignments.

I want you to use now and later in case of many valued semantics a more intelligent (and much faster!) method called **Proof by Contradiction Method**

In fact, I will **not accept** the Truth Tables verifications on any TEST and students using it will get **0 pts** for the problem.
Tautology - Proof by Contradiction Method

**Proof by Contradiction Method:**

In this method, in order to **prove** that $\models A$ we proceed as follows.

We **assume** that $\not\models A$

We **work** with this assumption.

If we get a **contradiction**, we have **proved** that $\not\models A$ is impossible.

We hence **proved** $\models A$.

If we do not get a **contradiction**, it means that the assumption $\not\models A$ is **true**, i.e.

we have **proved** that $\not\models A$. 
Tautology - Proof by Contradiction Method

**Proof by Contradiction Method:**

In order to verify whether $\models A$ one works backwards, trying to find a truth assignment $v$ which makes a formula $A$ false. If we find one, it means that $A$ is not a tautology if we prove that it is impossible, i.e. we got a contradiction. It means that the formula is a tautology.
Example

Let \( A = (a \Rightarrow (a \cup b)) \)

**Step 1:** Assume that \( \not\models A \), i.e. we write in a shorthand notion \( A = F \)

**Step 2:** We use shorthand notation to analyze **Strep 1**
\[(a \Rightarrow (a \cup b)) = F \text{ iff } a = T \text{ and } (a \cup b) = F\]

**Step 3:** Analyze **Step 2**
\[a = T \text{ and } (a \cup b) = F, \text{ i.e. } (T \cup b) = F\]

This is **impossible** by the definition of \( \cup \)
We got a **contradiction**, hence

\[\models (a \Rightarrow (a \cup b))\]
Example

Observe that exactly the same reasoning proves that for any formulas $A, B \in \mathcal{F}$,

$$\models (A \Rightarrow (A \cup B))$$

The following formulas are also tautologies

$$(((a \Rightarrow b) \land \neg c) \Rightarrow (((a \Rightarrow b) \land \neg c) \lor \neg d)),$$

and

$$((((((a \Rightarrow b) \land \neg c) \lor d) \land \neg e) \Rightarrow ((((a \Rightarrow b) \land \neg c) \lor d) \land \neg e) \lor (a \Rightarrow \neg e)))$$

because they are particular cases of $$(A \Rightarrow (A \cup B))$$
Tautologies, Contradictions

Set of all Tautologies

\[ T = \{ A \in \mathcal{F} : \models A \} \]

Definition

A formula \( A \in \mathcal{F} \) is called a contradiction if it does not have a model

Contradiction Notation: \( \models \neg A \)

Directly from the definition we have that

\[ \models A \quad \text{if and only if} \quad \forall v : \mathbf{VAR} \rightarrow \{ T, F \} \]

Set of all Contradictions

\[ C = \{ A \in \mathcal{F} : \models \neg A \} \]
Examples

Tautology \((A \implies (B \implies A))\)
Contradiction \((A \cap \neg A)\)
Neither \((a \cup \neg b)\)

Consider the formula \((a \cup \neg b)\)
Any \(v\) such that \(v(a) = T\) is a model for \((a \cup \neg b)\), so it is not a contradiction
Any \(v\) such that \(v(a) = F, \ v(b) = T\) is a counter-model for \((a \cup \neg b)\) so \(\not\models (a \cup \neg b)\)
Simple Properties

**Theorem 1** For any formula $A \in \mathcal{F}$ the following conditions are equivalent.

1. $A \in T$
2. $\neg A \in C$
3. For all $v$, $v \models A$

**Theorem 2** For any formula $A \in \mathcal{F}$ the following conditions are equivalent.

1. $A \in C$
2. $\neg A \in T$
3. For all $v$, $v \not\models A$
Constructing New Tautologies

We now formulate and prove a theorem which describes validity of a method of constructing new tautologies from given tautologies.

First we introduce some convenient notations.

**Notation 1**: for any $A \in \mathcal{F}$ we write $A(a_1, a_2, ... a_n)$ to denote that $a_1, a_2, ... a_n$ are all propositional variables appearing in $A$.

**Notation 2**: let $A_1, ... A_n$ be any formulas, we write $A(a_1/A_1, ..., a_n/A_n)$ to denote the result of simultaneous replacement (substitution) all variables $a_1, a_2, ... a_n$ in $A$ by formulas $A_1, ... A_n$, respectively.
Constructing New Tautologies

**Theorem** For any formulas $A, A_1, ... A_n \in \mathcal{F}$,
if $\models A(a_1, a_2, ... a_n)$ and $B = A(a_1/A_1, ..., a_n/A_n)$,
then $\models B$

**Proof:** Let $B = A(a_1/A_1, ..., a_n/A_n)$ and let $b_1, b_2, ... b_m$ be all propositional variables which occur in $A_1, ... A_n$
Given a truth assignment $v : \text{VAR} \rightarrow \{T, F\}$, the values $v(b_1), v(b_2), ... v(b_m)$ define $v^*(A_1), ... v^*(A_n)$ and, in turn define $v^*(A(a_1/A_1, ..., a_n/A_n))$
Constructing New Tautologies

Let now \( w : \text{VAR} \rightarrow \{ T, F \} \) be a truth assignment such that 
\( w(a_1) = v^*(A_1), \ w(a_2) = v^*(A_2), \ldots w(a_n) = v^*(A_n). \)

Obviously, \( v^*(B) = w^*(A). \)

Since \( \models A, \ w^*(A) = T, \) for all possible \( w, \)

hence \( v^*(B) = w^*(A) = T \) for all truth assignments \( w \) and

we have \( \models B \)
Models for Sets of Formulas

Consider \( \mathcal{L} = \mathcal{L}_{\text{CON}} \) and let \( S \neq \emptyset \) be any non empty set of formulas of \( \mathcal{L} \), i.e.

\[
S \subseteq F
\]

We adopt the following definition.

**Definition**

A truth assignment \( v : \text{VAR} \rightarrow \{T, F\} \) is a **model for the set** \( S \) of formulas if and only if

\[
v \models A \quad \text{for all formulas } A \in S
\]

We write

\[
v \models S
\]

to denote that \( v \text{ is a model for the set } S \) of formulas.
Similarly, we define a notion of a **counter-model**

**Definition**
A truth assignment $v : \text{VAR} \to \{T, F\}$ is a **counter-model for the set** $S \neq \emptyset$ of formulas if and only if

$$v \not\models A$$

for some formula $A \in S$

We write

$$v \not\models S$$

to denote that $v$ is a **counter-model** for the set $S$ of formulas
Remark that the set $S$ can be infinite, or finite
In a case when $S$ is a finite subset of formulas we define, as before, a notion of restricted model and restricted counter-model.

Definition
Let $S$ be a finite subset of formulas and $v \models S$ Any restriction of the model $v$ to the domain

$$VAR_S = \bigcup_{A \in S} VAR_A$$

is called a restricted model for $S$
Restricted Counter - Model for Sets of Formulas

Definition
Any restriction of a counter-model \( v \) of a set \( S \neq \emptyset \) of formulas to the domain

\[
VAR_S = \bigcup_{A \in S} VAR_A
\]

is called a restricted counter-model for \( S \)
Example

Example

Let $\mathcal{L} = \mathcal{L}_{\neg, \cap}$ and let

$$S = \{a, (a \cap \neg b), c, \neg b\}$$

We have now $\text{VAR}_S = \{a, b, c\}$ and $v : \text{VAR}_S \to \{T, F\}$ such that $v(a) = T, v(c) = T, v(b) = F$ is a restricted model for $S$ and $v : \text{VAR}_S \to \{T, F\}$ such that $v(a) = F$ is a restricted counter-model for $S$.
The set \( S \) from the previous example was a finite set.

Natural question arises:

**Question**

Give an example of an infinite set \( S \) that **has a model**

Give an example of an infinite set \( S \) that **does not have model**

**Ex1** Consider set \( T \) of all tautologies

It is a countably infinite set and by definition of a tautology any \( v \) is a model for \( T \), i.e. \( v \models T \)

**Ex2** Consider set \( C \) of all contradictions

It is a countably infinite set and

for any \( v, \ v \not\models C \) by definition of a contradiction, i.e. any any \( v \) is a counter-model for \( C \)
Challenge Problems

P1  Give an example of an infinite set $S$, such that $S \neq T$ and $S$ has a model

P2  Give an example of an infinite set $S$, such that $S \cap T = \emptyset$ and $S$ has a model

P3  Give an example of an infinite set $S$, such that $S \neq C$ and $S$ does not have a model

P4  Give an example of an infinite set $S$, such that $S \neq C$ and $S$ has a counter model

P5  Give an example of an infinite set $S$, such that $S \cap C = \emptyset$ and $S$ has a counter model
Chapter 4: Consistent Sets of Formulas

Definition
A set $G \subseteq F$ of formulas is called consistent if and only if $G$ has a model, i.e. we have that

$G \subseteq F$ is consistent if and only if

there is $v$ such that $v \models G$

Otherwise $G$ is called inconsistent
HALF Challenge Problems

P6 Give an example of an infinite set \( S \), such that \( S \neq T \) and \( S \) is consistent

P7 Give an example of an infinite set \( S \), such that \( S \cap T = \emptyset \) and \( S \) is consistent

P8 Give an example of an infinite set \( S \), such that \( S \neq C \) and \( S \) is inconsistent

P9 Give an example of an infinite set \( S \), such that \( S \cap C = \emptyset \) and \( S \) is inconsistent
Chapter 4: Independent Statements

Definition

A formula $A$ is called independent from a set $G \subseteq \mathcal{F}$ if and only if there are truth assignments $v_1, v_2$ such that

$$v_1 \models G \cup \{A\} \text{ and } v_2 \models G \cup \{\neg A\}$$

i.e. we say that a formula $A$ is independent if and only if

$G \cup \{A\}$ and $G \cup \{\neg A\}$ are consistent
Example

Example
Given a set

\[ G = \{ ((a \land b) \Rightarrow b), (a \lor b), \neg a \} \]

Show that \( G \) is consistent

Solution
We have to find \( v : VAR \to \{ T, F \} \) such that

\[ v \models G \]

It means that we need to find \( v \) such that

\[ v^*((a \land b) \Rightarrow b) = T, \quad v^*(a \lor b) = T, \quad v^*(\neg a) = T \]
Consistent: Example

1. Formula \(((a \cap b) \Rightarrow b)\) is a tautology, i.e. 
   \(v^*((a \cap b) \Rightarrow b) = T\) for any \(v\) and we do not need to consider it anymore.

2. Formula \(\neg a = T\) (we use shorthand notation) if and only if \(a = F\) so we get that \(v\) must be such that \(v(a) = F\)

2. We want \((a \cup b) = T\) but \(v\) is such that \(v(a) = F\) so \((a \cup b) = F \cup b = T\) if and only if \(b = T\)

This means that for any \(v : VAR \rightarrow \{T, F\}\) such that 
\(v(a) = F,\ v(b) = T\)

\(v \models G\)

and we proved that \(G\) is consistent
Independent: Example

Example
Show that a formula \( A = ((a \Rightarrow b) \cap c) \) is independent of \( G = \{((a \cap b) \Rightarrow b), (a \cup b), \neg a\} \)

Solution
We construct \( v_1, v_2 : VAR \rightarrow \{T, F\} \) such that

\[ v_1 \models G \cup \{A\} \quad \text{and} \quad v_2 \models G \cup \{\neg A\} \]

We have just proved that any \( v : VAR \rightarrow \{T, F\} \) such that \( v(a) = F, \ v(b) = T \) is a model for \( G \)
Independent: Example

Take as \( v_1 \) any truth assignment such that
\[ v_1(a) = v(a) = F, \quad v_1(b) = v(b) = T, \quad v_1(c) = T \]
We evaluate \( v_1^*(A) = v_1^*((a \Rightarrow b) \cap c) = (F \Rightarrow T) \cap T = T \)
This proves that \( v_1 \models G \cup \{A\} \)

Take as \( v_2 \) any truth assignment such that
\[ v_2(a) = v(a) = F, \quad v_2(b) = v(b) = T, \quad v_2(c) = F \]
We evaluate \( v_2^*(\neg A) = v_2^*(\neg(a \Rightarrow b) \cap c)) = T \cap T = T \)
This proves that \( v_2 \models G \cup \{\neg A\} \)

It ends the proof that \( A \) is independent of \( G \)
Not Independent: Example

Example
Show that a formula $A = (\neg a \land b)$ is not independent of

$$G = \{((a \land b) \Rightarrow b), (a \lor b), \neg a\}$$

Solution
We have to show that it is impossible to construct $v_1, v_2$ such that

$$v_1 \models G \cup \{A\} \quad \text{and} \quad v_2 \models G \cup \{\neg A\}$$

Observe that we have just proved that any $v$ such that $v(a) = F$, and $v(b) = T$ is the only model restricted to the set of variables $\{a, b\}$ for $G$ and $\{a, b\} = VAR_A$

So we have to check now if it is possible $v \models A$ and $v \models \neg A$
Not Independent: Example

We have to evaluate $v^*(A)$ and $v^*(\neg A)$ for $v(a) = F$, and $v(b) = T$

$v^*(A) = v^*((\neg a \land b) = \neg v(a) \land v(b) = \neg F \land T = T \land T = T$

and so $v \models A$

$v^*(\neg A) = \neg v^*(A) = \neg T = F$

and so $v \not\models \neg A$

This end the proof that $A$ is not independent of $G$
Example

Given a set \( G = \{a, (a \Rightarrow b)\} \), find a formula \( A \) that is independent from \( G \).

Observe that \( v \) such that \( v(a) = T, v(b) = T \) is the only restricted model for \( G \).

So we have to come up with a formula \( A \) such that there are two different truth assignments, \( v_1 \) and \( v_2 \), and

\[
v_1 \models G \cup \{A\} \quad \text{and} \quad v_2 \models G \cup \{\neg A\}
\]

Let’s consider \( A = c \), then \( G \cup \{A\} = \{a, (a \Rightarrow b), c\} \).

A truth assignment \( v_1 \), such that \( v_1(a) = T, v_1(b) = T \) and \( v_1(c) = T \) is a model for \( G \cup \{A\} \).

Likewise for \( G \cup \{\neg A\} = \{a, (a \Rightarrow b), \neg c\} \).

Any \( v_2 \), such that \( v_2(a) = T, v_2(b) = T \) and \( v_2(c) = F \) is a model for \( G \cup \{\neg A\} \) and so the formula \( A \) is independent.
Challenge Problem

Find an infinite number of formulas that are independent of a set

\[ G = \{((a \cap b) \Rightarrow b), (a \cup b), \neg a\} \]
Challenge Problem Solution

This my solution - there are many others- this one seemed to me the **most simple**

**Solution**

We just proved that any $v$ such that $v(a) = F, \ v(b) = T$ is the **only** model restricted to the set of variables $\{a, b\}$ and so all other possible models for $G$ must be **extensions** of $v$
We define a countably infinite set of formulas (and their negations) and corresponding extensions of \( v \) (restricted to the set of variables \( \{a, b\} \) such that \( v \models G \) as follows.

Observe that all extensions of \( v \) restricted to the set of variables \( \{a, b\} \) have as domain the infinitely countable set

\[
\text{VAR} = \{a_1, a_2, \ldots, a_n, \ldots\}
\]

We take as an infinite set of formulas in which every formula independent of \( G \) the set of atomic formulas

\[
\mathcal{F}_0 = \{a_1, a_2, \ldots, a_n, \ldots\} - \{a, b\}
\]
Challenge Problem Solution

Let \( c \in F_0 = \{a_1, a_2, \ldots, a_n, \ldots\} - \{a, b\} \)

We define truth assignments \( v_1, v_2 : VAR \rightarrow \{T, F\} \) such that

\[
v_1 \models G \cup \{c\} \quad \text{and} \quad v_2 \models G \cup \{\neg c\}
\]

as follows

\[
v_1(a) = v(a) = F, \quad v_1(b) = v(b) = T \quad \text{and} \quad v_1(c) = T \quad \text{for any} \quad c \in F_0
\]

\[
v_2(a) = v(a) = F, \quad v_2(b) = v(b) = T \quad \text{and} \quad v_2(c) = F \quad \text{for any} \quad c \in F_0
\]