QUESTION 1 (5pts)

Please write carefully your solutions. NO PARTIAL CREDIT. Formulas must be fully correct for credit.

Write the following natural language statement:

It is possible that one likes when it is cold and from the fact that it is necessary to wear a scarf, we conclude the following: one does not like when it is cold or one likes when it is not necessary to wear a scarf

in the following two ways.

1. (3pts) Formula $A_1 \in F_1$ of a language $L_{\{\neg, \land, \lor, \Rightarrow\}}$, where $LA$ represents statement "one likes A", and $\Box, \Diamond$ are modal connectives of necessity, possibility, respectively.

Solution We translate our statement into a formula $A_1 \in F_1$ of a language $L_{\{\neg, \land, \lor, \Rightarrow\}}$ as follows.

Propositional Variables: $a, b$

Translation 1

$A_1 = (\Diamond LA \land (\Box b \Rightarrow (\neg I a \lor L \neg \Box b)))$

2. (2pts) As a formula $A_2 \in F_2$ of a language $L_{\{\neg, \land, \lor, \Rightarrow\}}$.

Solution We translate our statement into a formula $A_2 \in F_2$ of a language $L_{\{\neg, \land, \lor, \Rightarrow\}}$ as follows.

Propositional Variables: $a, b, c, d$

Translation 2:

$A_2 = (a \land (b \Rightarrow (\neg c \lor d)))$
QUESTION 2  (5pts)

Please write carefully your solutions. NO PARTIAL CREDIT. Formulas must be fully correct for credit.

Here is a mathematical statement $S$:

For all rational numbers $x \in Q$ the following holds: If $x = 0$, then there is a natural number $n \in N$, such that $x + n = 0$

1. (2pts) Re-write $S$ as a symbolic mathematical statement $SM$ that only uses mathematical and logical symbols.

Solution

$S$ becomes a symbolic mathematical statement

$$SM : \forall x \in Q (x = 0 \Rightarrow \exists n \in N x + n = 0)$$

2. (2pts) Translate the symbolic statement $SM$ into a corresponding formula with restricted quantifiers. Explain your choice of symbols.

Solution

We write $Q(x)$ for $x \in Q$, $N(y)$ for $y \in N$, a constant $c$ for the number $0$.

We use $E \in P$ to denote the relation $=$, we use $f \in F$ to denote the function $+$.

The statement $x = 0$ becomes an atomic formula $E(x, c)$.

The statement $x + n = 0$ becomes an atomic formula $E(f(x, y), c)$.

The symbolic mathematical statement $SM$ becomes a restricted quantifiers formula

$$\forall x (Q(x) \Rightarrow (E(x, c) \Rightarrow \exists y (N(y) \cap E(f(x, y), c))))$$

3. (1pts) Translate your restricted domain quantifiers formula into a correct formula $A$ of the predicate language $L$.

Solution

We apply now the transformation rules and get a corresponding formula $A \in F$:

$$\forall x (Q(x) \Rightarrow (E(x, c) \Rightarrow \exists y ((N(y) \cap E(f(x, y), c))))$$
QUESTION 3 (5pts)

1. (3pts)

Circle formulas that are propositional tautologies

\[ S_1 = \{ (a \land \neg c) \Rightarrow ((a \land \neg c) \lor (\neg b \Rightarrow a)), \quad ((a \Rightarrow b) \lor (a \land \neg b)), \quad (A \lor (A \Rightarrow \neg A)), \quad (a \lor \neg b) \} \]

Solution

The formula \((a \lor \neg b)\) is the only formula that is not a tautology.

2. (2pts)

Circle formulas that are predicate tautologies

\[ S_2 = \{ (\forall x (R(x) \Rightarrow \neg R(x))) \Rightarrow \exists x (R(x) \Rightarrow \neg R(x)) \}, \]

\[ (\forall x (\neg P(x,y) \land P(x,y)) \Rightarrow \exists x P(x,y)), \quad ((\exists x A(x) \land \exists x B(x)) \Rightarrow \exists x (A(x) \land B(x))) \}

Solution

The formula \((\exists x A(x) \land \exists x B(x)) \Rightarrow \exists x (A(x) \land B(x)))\) is the only formula that is not a tautology.

QUESTION 4 (5pts)

Given a formula \(A : \forall x \exists y P(f(x,y),c)\) of the predicate language \(L\), and

two model structures \(M_1 = (Z, I_1)\), \(M_2 = (N, I_2)\) with the interpretations defined as follows.

\(P_{I_1} := \), \(f_{I_1} := +\), \(c_{I_1} := 0\) and \(P_{I_2} := >\), \(f_{I_2} := \cdot\), \(c_{I_2} := 0\).

1. (2.5 pts) Show that \(M_1 \models A\)

Solution

\(A_{I_1} : \forall_{x \in Z} \exists_{y \in Z} x + y = 0\) is a true statement;

For each \(x \in Z\) exists \(y = -x\) and \(-x \in Z\) and \(x - x = 0\).

2. (2.5 pts) Show that \(M_2 \not\models A\)

Solution

\(A_{I_2} : \forall_{x \in N} \exists_{y \in N} x \cdot y > 0\) is a false statement for \(x = 0\).
**QUESTION 5 (5pts)**

Given a $M$ semantics for $L_{\{\neg, \land, \lor, \Rightarrow\}}$, where the connectives are defined as follows:

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Given a formulas ($(a \land b) \Rightarrow \neg b$) and $(((b \Rightarrow \neg a) \Rightarrow (a \Rightarrow \neg b)) \cup (a \Rightarrow b))$

Use the **fact** that a truth assignment $v : VAR \rightarrow \{F, \perp, T\}$ is such that

$v^\prime((a \land b) \Rightarrow \neg b) = \perp$

under $M$ semantics to evaluate $v^\prime(((b \Rightarrow \neg a) \Rightarrow (a \Rightarrow \neg b)) \cup (a \Rightarrow b))$

Use shorthand notation.

**Solution**

$((a \land b) \Rightarrow \neg b) = \perp$ in two cases.

**C1** $(a \land b) = \perp$ and $\neg b = F$.

**C2** $(a \land b) = T$ and $\neg b = \perp$.

Consider Case C1

$\neg b = F$, i.e. $b = T$, and hence $(a \land T) = \perp$ if and only if $a = \perp$.

We get that $v$ is such that $v(a) = \perp$ and $v(b) = T$.

We evaluate:

$v^\prime(((b \Rightarrow \neg a) \Rightarrow (a \Rightarrow \neg b)) \cup (a \Rightarrow b)) = (((T \Rightarrow \neg \perp) \Rightarrow (\perp \Rightarrow \neg T)) \cup (\perp \Rightarrow T)) = ((\perp \Rightarrow \perp) \cup T) = T$

Consider Case C2

$\neg b = \perp$, i.e. $b = T$, and hence $(a \land \perp) = T$ which is impossible, hence $v$ from case C1 is the only one.