QUESTION 1  (20pts)
Write the following natural language statement:

One likes to play bridge, or from the fact that the weather is good we conclude the following: one does not like to play bridge or one likes not to play bridge

as a formula of 2 different languages

1. Formula $A_1 \in F_1$ of a language $L_{\{\neg, L, \cup, \Rightarrow\}}$, where $LA$ represents statement ”one likes A”, ”A is liked”.

Solution  We translate our statement into a formula $A_1 \in F_1$ of a language $L_{\{\neg, L, \cup, \Rightarrow\}}$ as follows.

Propositional Variables: $a, b$

Translation 1
$A_1 = (La \cup (b \Rightarrow (\neg I a \cup L \neg a)))$

2. Formula $A_2 \in F_2$ of a language $L_{\{\neg, \cup, \Rightarrow\}}$.

Solution  We translate our statement into a formula $A_2 \in F_2$ of a language $L_{\{\neg, \cup, \Rightarrow\}}$ as follows.

Propositional Variables: $a, b, c$

Translation 2:
$A_2 = (a \cup (b \Rightarrow (\neg a \cup c)))$

QUESTION 2  (20pts)
Let $A$ be a formula $(((a \cap \neg c) \Rightarrow \neg b) \cup a) \Rightarrow (c \cup b))$.

1. (5pts) A language $L_{CON}$ to which the formula $A$ belongs is:

Solution: The language is $L_{\{\neg, \cap, \Rightarrow\}}$.

2. (5pts) Determine the degree of $A$ and write down all its sub-formulas of the degree 2.

Solution: The degree of $A$ is 7. There is only one sub-formula of the degree 2: $(a \cap \neg c)$.

3. (5pts) Determine whether : $A \in T$. Use ”proof by contradiction” method and shorthand notation.

Solution: of the case $A \in T$.
Assume $(((a \cap \neg c) \Rightarrow \neg b) \cup a) \Rightarrow (c \cup b)) = F$. This is possible if and only if $((a \cap \neg c) \Rightarrow \neg b) \cup a) = T$ and $(c \cup b) = F$. This gives as that $c = F, b = F$. We evaluate $((a \cap \neg F) \Rightarrow \neg F) \cup a) = T$. This is possible for $a = T$.
Any truth assignment such that $a = T, b = F, c = F$ is a counter-model for $A$, hence $A \notin T$. 

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4. (5pts) Determine whether \( A \in C \).

**Solution:** any truth assignment such that \( a = T, b = T, c = F \) is a model for \( A \), hence \( A \not\in C \). This is not the only model.

**QUESTION 3 (20 pts)**

1. (10pts) We define: A set \( G \subseteq F \) is **consistent** if and only if there is a truth assignment \( v \) such that \( v \models G \).

**Prove** that the set \( G \) below is consistent. Use shorthand notation.

\[
G = \left\{ (a \Rightarrow (a \cup b)), \ (a \cup b), \ \neg b, \ (c \Rightarrow b) \right\}
\]

**Solution:** We find a restricted model for \( G \) as followas

First observe that the formula \((a \Rightarrow (a \cup b))\), is a tautology, hence any \( v \) is its model. So we have only to see whether two other formulas have a common model. It means we check if it is possible to find \( v \), such that \( v^* (\neg b) = T \), \( v^* ((a \cup b)) = T \), and \( v^* ((c \Rightarrow b)) = T \).

We have that \( \neg b = T \) if and only if \( b = F \).

We evaluate \((a \cup b) = (a \cup F) = T \) if and only if \( a = T \).

Consequently, \((c \Rightarrow b) = (c \Rightarrow F) = T \) if and only if \( c = F \).

Hence, any \( v \), such that \( a = T, b = T, \) and \( c = F \) is a model for \( G \).

**Observe** that \( a = T, b = T, \) and \( c = F \) is the only restricted model for \( G \).

2. (10pts) We define: a formula \( A \in F \) is called **independent** from a set \( G \subseteq F \) if and only if the sets \( G \cup \{ A \} \) and \( G \cup \{ \neg A \} \) are both consistent.

I.e. when there are truth assignments \( v_1, v_2 \) such that \( v_1 \models G \cup \{ A \} \) and \( v_2 \models G \cup \{ \neg A \} \).

**FIND** an infinite number of formulas that are independent of a set \( G \). Use shorthand notation.

**Solution:** We know that \( a = T, b = T, \) and \( c = F \) is the only restricted model for \( G \).

Let \( A \) be any atomic formula \( d \), where \( d \in VAR - \{a, b, c\} \).

Any \( v \), such that \( a = T, b = T, c = F, \) and \( d = T \) is a model for \( G \cup \{ d \} \).

Any \( v \), such that \( a = T, b = T, c = F, \) and \( d = F \) is a model for \( G \cup \{ \neg d \} \).

There is countably infinitely many atomic formulas \( A = d \), where \( d \in VAR - \{a, b, c\} \).

**QUESTION 4 (15 pts)**

We define a 3 valued extensional semantics \( M \) for the language \( L_\{\neg, \cup, \Rightarrow\} \) by defining the connectives \( \neg, \cup, \Rightarrow \) on a set \( \{F, \bot, T\} \) of logical values by the following truth tables.
1. (5pts) Verify whether \( \models_M (L \cup -L) \). You can use shorthand notation.

**Solution**

We verify

\[
\begin{array}{c|ccc}
L & F & \bot & T \\
\hline
F & F & F & T \\
\end{array}
\]

2. (5pts) Verify whether your formulas \( A_1 \) and \( A_2 \) from QUESTION 1 have a model/counter model under the semantics \( M \). You can use shorthand notation.

**Solution**

The formulas are: \( A_1 = (L \cup (b \Rightarrow (¬L \cup ¬L))) \), and \( A_2 = (a \cup (b \Rightarrow (¬a \cup c))) \).

Any \( v \), such that \( v(a) = T \) is a \( M \) model for \( A_1 \) and for \( A_2 \) directly from the definition of \( ∪ \).

3. (5pts) Verify whether the following set \( G \) is \( M \)-consistent. You can use shorthand notation

\[
G = \{ L_a, (a \cup -L_b), (a \Rightarrow b), b \}
\]

**Solution**

Any \( v \), such that \( v(a) = T, v(b) = T \) is a \( M \) model for \( G \) as

\[
\begin{array}{c|ccc}
L & F & \bot & T \\
\hline
F & F & F & T \\
\end{array}
\]

\[
\begin{array}{c|ccc}
\Rightarrow & F & \bot & T \\
\hline
F & T & T & T \\
\end{array}
\]

\[
\begin{array}{c|ccc}
\cup & F & \bot & T \\
\hline
F & F & F & T \\
\end{array}
\]

**QUESTION 5 (15pts)**

1. (10pts) Given a formula \( A = ((a \cap ¬c) \Rightarrow (¬a \cup b)) \) of a language \( L_{\{¬, \cap, \cup, \Rightarrow\}} \).

Find a formula \( B \) of a language \( L_{\{¬, \Rightarrow\}} \), such that \( A \equiv B \). List all proper logical equivalences used at each step.

**Solution**

\[
A = ((a \cap ¬c) \Rightarrow (¬a \cup b)) \equiv ((a \cap ¬c) \Rightarrow (a \Rightarrow b)) \equiv (¬(a \Rightarrow ¬c) \Rightarrow (a \Rightarrow b)) \equiv (¬(a \Rightarrow c) \Rightarrow (a \Rightarrow b)) = B
\]

Equivalences used: 1. \( (¬A \cup B) \equiv (A \Rightarrow B) \), 2. \( (A \cap B) \equiv ¬(A \Rightarrow ¬B) \), 3. \( ¬¬A \equiv A \).
2. (5pts) Prove that \( L_{\{\neg, \cap, \cup, \Rightarrow\}} \equiv L_{\{\neg, \Rightarrow\}} \)

**Solution** We have to prove that \( L_{\{\neg, \Rightarrow\}} \equiv L_{\{\neg, \cap, \cup, \Rightarrow\}} \).

Condition **C1** holds because \( \{\neg, \Rightarrow\} \subseteq \{\neg, \cap, \cup, \Rightarrow\} \).

Condition **C2** holds because of the **Substitution Theorem** and because of the following logical equivalences for any formulas \( A, B \):

\[
(A \cap B) \equiv \neg(A \Rightarrow \neg B) \quad \text{and} \quad (A \cup B) \equiv (\neg A \Rightarrow B)
\]

**Reminder** We define the **equivalence of languages** as follows:

Given two languages: \( L_1 = L_{CON_1} \) and \( L_2 = L_{CON_2} \), for \( CON_1 \neq CON_2 \), we say that they are **logically equivalent**, i.e. \( L_1 \equiv L_2 \) if and only if the following conditions **C1**, **C2** hold.

**C1:** For every formula \( A \) of \( L_1 \), there is a formula \( B \) of \( L_2 \), such that \( A \equiv B \).

**C2:** For every formula \( C \) of \( L_2 \), there is a formula \( D \) of \( L_1 \), such that \( C \equiv D \).