CSE371 Midterm Solutions Spring 2020

QUESTION 1 (20pts)

Write the following natural language statement:

> **One likes to play bridge, or from the fact that the weather is good we conclude the following: one does not like to play bridge or one likes not to play bridge**
>
> as a formula of 2 different languages

1. Formula $A_1 \in \mathcal{F}_1$ of a language $\mathcal{L}_{\{\neg,\ \mathbf{L},\ \cup,\ \Rightarrow\}}$, where $\mathbf{L}A$ represents statement "one likes A", "A is liked".

Solution We translate our statement into a formula
$A_1 \in \mathcal{F}_1$ of a language $\mathcal{L}_{\{\neg,\ \mathbf{L},\ \cup,\ \Rightarrow\}}$ as follows.

Propositional Variables: a, b
a denotes statement: *play bridge*,
b denotes a statement: *the weather is good*

Translation 1

$$A_1 = (\mathbf{L}a \cup (b \Rightarrow (\neg \mathbf{I}a \cup \mathbf{L}\neg a)))$$

2. Formula $A_2 \in \mathcal{F}_2$ of a language $\mathcal{L}_{\{\neg,\ \cup,\ \Rightarrow\}}$.

Solution We translate our statement into a formula $A_2 \in \mathcal{F}_2$ of a language $\mathcal{L}_{\{\neg,\ \cup,\ \Rightarrow\}}$ as follows.

Propositional Variables: a, b, c
a denotes statement: *One likes to play bridge* ,
b denotes a statement: *the weather is good*,
c denotes a statement: *one likes not to play bridge*

Translation 2:

$$A_2 = (a \cup (b \Rightarrow (\neg a \cup c)))$$

QUESTION 2 (20pts)

Let A be a formula $((((a \cap \neg c) \Rightarrow \neg b) \cup a) \Rightarrow (c \cup b))$.

1. (5pts) A language $\mathcal{L}_{CON}$ to which the formula A belongs is:

Solution: The language is $\mathcal{L}_{\{\neg, \cap, \Rightarrow\}}$.

2. (5pts) Determine the degree of A and write down all its sub-formulas of the degree 2.

Solution: The degree of A is 7. There is only one sub-formula of the degree 2: $(a \cap \neg c)$.

3. (5pts) Determine whether : $A \in \mathbf{T}$. Use "proof by contradiction" method and **shorthand** notation.

Solution: of the case $A \in \mathbf{T}$.
Assume $((((a \cap \neg c) \Rightarrow \neg b) \cup a) \Rightarrow (c \cup b)) = F$. This is possible if and only if $(((a \cap \neg c) \Rightarrow \neg b) \cup a) = T$ and $(c \cup b) = F$. This gives as that $c = F, b = F$. We evaluate $(((a \cap \neg F) \Rightarrow \neg F) \cup a) = T$. This is possible for $a = T$.
Any truth assignment such that $a = T, b = F, c = F$ is a counter-model for A, hence $A \notin \mathbf{T}$.

4. (5pts) Determine whether $A \in \mathbf{C}$.

Solution: any truth assignment such that $a = T,\ b = T,\ c = F$ is a model for A, hence $A \notin \mathbf{C}$. This is not the only model.

QUESTION 3 (20 pts)

1. (10pts) We define: A set $\mathcal{G} \subseteq \mathcal{F}$ is **consistent** if and only if there is a truth assignment v such that $v \models \mathcal{G}$

Prove that the set $\mathcal{G}$ below is **consistent**. Use **shorthand** notation.

$$\mathcal{G} = \{(a \Rightarrow (a \cup b)),\ (a \cup b),\ \neg b,\ (c \Rightarrow b)\}$$

Solution: We find a restricted model for $\mathcal{G}$ as followas

First observe that the formula $((a \Rightarrow a \cup b))$, is a tautology, hence any v is its model. So we have only to see whether two other formulas have a common model. It means we check if it is possible to find v, such that $v^*(\neg b) = T$, $v^*((a \cup b)) = T$, and $v^*((c \Rightarrow b)) = T$.

We have that $\neg b = T$ if and only if $b = F$.

We evaluate $(a \cup b) = (a \cup F) = T$ if and only if $a = T$.

Consequently, $(c \Rightarrow b) = (c \Rightarrow F) = T$ if and only if $c = F$.

Hence, any v, such that $a = T,\ b = T,$ and $c = F$ is a model for $\mathcal{G}$.

Observe that $a = T,\ b = T,$ and $c = F$ is the only restricted model for $\mathcal{G}$.

2. (10pts) We define: a formula $A \in \mathcal{F}$ is called **independent** from a set $\mathcal{G} \subseteq \mathcal{F}$ if and only if

the sets $\mathcal{G} \cup \{A\}$ and $\mathcal{G} \cup \{\neg A\}$ are both **consistent**.

I.e. when there are truth assignments $v_1,\ v_2$ such that $v_1 \models \mathcal{G} \cup \{A\}$ and $v_2 \models \mathcal{G} \cup \{\neg A\}$.

FIND an infinite number of formulas that are **independent** of a set $\mathcal{G}$. Use shorthand notation.

Solution: We know that $a = T,\ b = T,\ and c = F$ is the only restricted model for $\mathcal{G}$.

Let A be any **atomic formula** d, where $d \in VAR - \{a, b, c\}$.

Any v, such that $a = T,\ b = T,\ c = F, and\ d = T$ is a **model** for $\mathcal{G} \cup \{d\}$.

Any v, such that $a = T,\ b = T,\ c = F, and\ d = F$ is a **model** for $\mathcal{G} \cup \{\neg d\}$.

There is countably infinitely many atomic formulas $A = d$, where $d \in VAR - \{a, b, c\}$.

QUESTION 4 (15 pts)

We define a 3 valued extensional semantics $\mathbf{M}$ for the language $\mathcal{L}_{\{\neg,\ \mathbf{L},\ \cup,\ \Rightarrow\}}$ by **defining the connectives** $\neg,\ \cup,\ \Rightarrow$ on a set $\{F, \perp, T\}$ of logical values by the following truth tables.

L Connective

$\mathbf{L}$	F	$\perp$	T
	F	F	T

Negation :

$\neg$	F	$\perp$	T
	T	F	F

Implication

$\Rightarrow$	F	$\perp$	T
F	T	T	T
$\perp$	T	$\perp$	T
T	F	F	T

Disjunction :

$\cup$	F	$\perp$	T
F	F	$\perp$	T
$\perp$	$\perp$	T	T
T	T	T	T

1. (5pts) Verify whether $\models_{\mathbf{M}} (\mathbf{L}A \cup \neg \mathbf{L}A)$. You can use **shorthand notation**.

Solution

We verify

$\mathbf{L}T \cup \neg \mathbf{L}T = T \cup F = T$, $\mathbf{L} \perp \cup \neg \mathbf{L} \perp = F \cup \neg F = F \cup T = T$, $\mathbf{L}F \cup \neg \mathbf{L}F = F \cup \neg F = T$

2. (5pts) Verify whether your formulas A_1 and A_2 from QUESTION 1 have a model/ counter model under the semantics $\mathbf{M}$. You can use **shorthand notation**.

Solution

The formulas are: $A_1 = (\mathbf{L}a \cup (b \Rightarrow (\neg \mathbf{I}a \cup \mathbf{L}\neg a)))$, and $A_2 = (a \cup (b \Rightarrow (\neg a \cup c)))$.

Any v, such that $v(a) = T$ is a **M model** for A_1 and for A_2 directly from the definition of $\cup$.

3. (5pts) Verify whether the following set $\mathbf{G}$ is $\mathbf{M}$-consistent. You can use **shorthand notation**

$$\mathbf{G} = \{\ \mathbf{L}a,\ \ (a \cup \neg \mathbf{L}b),\ \ (a \Rightarrow b),\ b\ \}$$

Solution

Any v, such that $v(a) = T$, $v(b) = T$ is a **M model** for **G** as

$$\mathbf{L}T = T, \quad (T \cup \neg \mathbf{L}T) = T, \quad (T \Rightarrow T) = T, \quad b = T.$$

QUESTION 5 (15pts)

1. (10pts) Given a formula $A = ((a \cap \neg c) \Rightarrow (\neg a \cup b))$ of a language $\mathcal{L}_{\{\neg, \cap, \cup, \Rightarrow\}}$.

Find a formula B of a language $\mathcal{L}_{\{\neg, \Rightarrow\}}$, such that $A \equiv B$. **List** all proper logical equivalences used at at each step.

Solution :

$A = ((a \cap \neg c) \Rightarrow (\neg a \cup b)) \equiv ((a \cap \neg c) \Rightarrow (a \Rightarrow b)) \equiv (\neg(a \Rightarrow \neg\neg c) \Rightarrow (a \Rightarrow b)) \equiv (\neg(a \Rightarrow c) \Rightarrow (a \Rightarrow b)) = B$

Equivalences used: **1.** $(\neg A \cup B) \equiv (A \Rightarrow B)$, **2.** $(A \cap B) \equiv \neg(A \Rightarrow \neg B)$, **3.** $\neg\neg A \equiv A$.

2. (5pts) Prove that $\mathcal{L}_{\{\neg,\cap,\cup,\Rightarrow\}} \equiv \mathcal{L}_{\{\neg,\Rightarrow\}}$

Solution We have to prove that $\mathcal{L}_{\{\neg,\Rightarrow\}} \equiv \mathcal{L}_{\{\neg,\cap,\cup,\Rightarrow\}}$.

Condition **C1** holds because $\{\neg,\Rightarrow\} \subseteq \{\neg,\cap,\cup,\Rightarrow\}$.

Condition **C2** holds because of the **Substitution Theorem** and because of the following **logical equivalences** for any for any formulas A, B

$$(A \cap B) \equiv \neg(A \Rightarrow \neg B) \qquad \text{and} \qquad (A \cup B) \equiv (\neg A \Rightarrow B)$$

Reminder We define the **equivalence of languages** as follows:

Given two languages: $\mathcal{L}_1 = \mathcal{L}_{CON_1}$ and $\mathcal{L}_2 = \mathcal{L}_{CON_2}$, for $CON_1 \neq CON_2$, we say that they are **logically equivalent**, i.e. $\mathcal{L}_1 \equiv \mathcal{L}_2$ if and only if the following conditions **C1, C2** hold.

C1: For every formula A of $\mathcal{L}_1$, there is a formula B of $\mathcal{L}_2$, such that $A \equiv B$,

C2: For every formula C of $\mathcal{L}_2$, there is a formula D of $\mathcal{L}_1$, such that $C \equiv D$.