

cse541
LOGIC for Computer Science

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LECTURE 9c

Chapter 9
Hilbert Proof Systems
Completeness of Classical Predicate Logic

PART 4: Deduction Theorem

Deduction Theorem

In **mathematical** arguments, one often **assumes** a statement **A** on the assumption (hypothesis) of some other statement **B** and then **concludes** that we have **proved** the implication "if A, then B"

This reasoning is **justified** by the following theorem, called a **Deduction Theorem**

It was first **formulated** and **proved** for a certain Hilbert proof system **S** for the classical **propositional** logic by **Herbrand** in **1930** in a form stated as follows

Deduction Theorem

Deduction Theorem (Herbrand,1930)

For any formulas A, B of the language of a **propositional** proof system S ,

if $A \vdash_S B$ then $\vdash_S (A \Rightarrow B)$

In **chapter 5** we formulated and proved the following, more **genera**l version of the Herbrand Theorem for a **very simple** (two logical axioms and Modus Ponens) **propositional** proof system **H1**

Deduction Theorem

Deduction Theorem

For any subset Γ of the set of formulas \mathcal{F} of H_1 and for any formulas $A, B \in \mathcal{F}$,

$$\Gamma, A \vdash_{H_1} B \text{ if and only if } \Gamma \vdash_{H_1} (A \Rightarrow B)$$

In particular,

$$A \vdash_{H_1} B \text{ if and only if } \vdash_{H_1} (A \Rightarrow B)$$

A natural **question** arises:

does **deduction theorem** hold for the **predicate** logic in general and for its proof system **H** we defined here?.

Deduction Theorem

The **Deduction Theorem** **can not** be carried directly to the **predicate** logic, but it nevertheless **holds** with **some modifications**. Here is where the problem lays.

Fact

Given the proof system

$$\mathbf{H} = (\mathcal{L}(\mathbf{P}, \mathbf{F}, \mathbf{C}), \mathcal{F}, LA, \mathcal{R} = \{(MP), (G), (G1), (G2)\})$$

For any formula $A(x) \in \mathcal{F}$,

$$A(x) \vdash \forall xA(x)$$

but it is **not always** the case that

$$\vdash (A(x) \Rightarrow \forall xA(x))$$

Deduction Theorem

Proof

Obviously, $A(x) \vdash \forall xA(x)$ by Generalization rule (G)

Let now $A(x)$ be an atomic formula $P(x)$

By the **H Completeness Theorem**

$\vdash (P(x) \Rightarrow \forall xP(x))$ if and only if $\models (P(x) \Rightarrow \forall xP(x))$

Consider a structure

$$\mathcal{M} = [M, I]$$

where M contains at least two elements c and d

We define $P_I \subseteq M$ as a property that holds **only** for c , i.e.

$$P_I = \{c\}$$

Deduction Theorem

Take any assignment $s : VAR \rightarrow M$

Then $(\mathcal{M}, s) \models P(x)$ only when $s(x) = c$ for all $x \in VAR$

$\mathcal{M} = [M, I]$ is a **counter model** for $(P(x) \Rightarrow \forall x P(x))$

as we found s such $(\mathcal{M}, s) \models P(x)$ and obviously
 $(\mathcal{M}, s) \not\models \forall x P(x)$

We proved that $\not\models (P(x) \Rightarrow \forall x P(x))$

By the **H Completeness Theorem** this is equivalent to

$$\not\vdash (P(x) \Rightarrow \forall x P(x))$$

and the **Deduction Theorem fails** as

$$Px \vdash \forall x P(x)$$

Deduction Theorem

The **Fact** shows that the **problem** is with application of the **generalization** rule (G) to the formula $A \in \Gamma$

To handle this we introduce, after **Mendelson(1987)** the following notion

Deduction Theorem

Definition

Let A be one of formulas in Γ and let

$$(P) \quad B_1, B_2, \dots, B_n$$

be a proof (deduction) of B_n from Γ , together with **justification** at each step. We say that the formula

B_i **depends upon** A in the proof B_1, B_2, \dots, B_n

if and only if the following holds

(1) B_i is A and the **justification** for B_i is $B_i \in \Gamma$

or

(2) B_i is **justified** as direct consequence by **MP**

or

(G) of some preceding formulas in the proof sequence (P), where at **least one** of these preceding formulas **depends upon** A

Deduction Theorem

Example

Here is a proof (deduction)

$$B_1, B_2, \dots, B_5$$

showing that

$$A, (\forall xA \Rightarrow C) \vdash \forall xC$$

$$B_1 \quad A$$

Hyp

B_1 depends upon A

$$B_2 \quad \forall xA$$

$$B_1, (G)$$

B_2 depends upon A

$$B_3 \quad (\forall xA \Rightarrow C)$$

Hyp

B_3 depends upon $(\forall xA \Rightarrow C)$

Deduction Theorem

$B_3 \quad (\forall xA \Rightarrow C)$

Hyp

B_3 depends upon $(\forall xA \Rightarrow C)$

$B_4 \quad C$

MP on B_2, B_3

B_4 depends upon A and $(\forall xA \Rightarrow C)$

$B_5 \quad \forall xC$

(G)

B_4 depends upon A and $(\forall xA \Rightarrow C)$

Observe that the formulas A, C may, or may not have x as a free variable

Deduction Theorem

DT Lemma

If B **does not** depend upon A in a proof (deduction) showing that $\Gamma, A \vdash B$, then $\Gamma \vdash B$

Proof

Let

$$B_1, B_2, \dots, B_n = B$$

be a proof (deduction) of B from Γ, A ,
in which B **does not** depend upon A

We prove by **induction** over the length of the proof that

$$\Gamma \vdash B$$

Deduction Theorem

Assume that **DT Lemma** holds for all proofs of the length less than n

If $B \in \Gamma$ or $B \in LA$, by definition then $\Gamma \vdash B$

If B is a direct **consequence** of two **preceding** formulas, then, since B **does not** depend upon A , **neither do** these preceding formulas

By **inductive** hypothesis, these **preceding** formulas have a proof from Γ alone

Hence **so does** B , i.e.

$$\Gamma \vdash B$$

Now we are ready to **formulate** and **prove** the **Deduction Theorem** for predicate logic

Deduction Theorem

Deduction Theorem

For any formulas A, B of the language of proof system H the following holds

(1) **Assume** that **in some** proof (deduction) showing that

$$\Gamma, A \vdash B$$

no application of the generalization rule (G) **to** a formula that **depends** upon A has as its **quantified** variable a **free** variable of the formula A

Then we have that

$$\Gamma \vdash (A \Rightarrow B)$$

(2) If $\Gamma \vdash (A \Rightarrow B)$, then $\Gamma, A \vdash B$

Deduction Theorem

Proof

The proof we present **extends** the proof of the **Deduction Theorem** for **propositional** logic from chapter 5

We **adopt** the **propositional proof** to the system **H** and add the relevant **predicate** cases

For the sake of **clarity** and **independence** we write now the **whole proof** in all **details**

Deduction Theorem

(1) Assume that

$$\Gamma, A \vdash B$$

i.e. that we have a formal proof

$$B_1, B_2, \dots, B_n$$

of B from the set of formulas $\Gamma \cup \{A\}$

In order to prove that

$$\Gamma \vdash (A \Rightarrow B)$$

we will prove the following a **stronger** statement

(S) $\Gamma \vdash (A \Rightarrow B_i)$ for all B_i ($1 \leq i \leq n$) in the proof of B

Deduction Theorem

Hence, in particular case, when $i = n$, we will obtain that also

$$\Gamma \vdash (A \Rightarrow B)$$

The proof of the statement **(S)** is conducted by **induction** on $1 \leq i \leq n$

Base Step $i = 1$

When $i = 1$, it means that the formal proof contains only one element B_1

By the definition of the formal proof from $\Gamma \cup \{A\}$, we have that $B_1 \in LA$, or $B_1 \in \Gamma$, or $B_1 = A$, i.e.

$$B_1 \in LA \cup \Gamma \cup \{A\}$$

Here we have **two** cases

Deduction Theorem

Case 1 $B_1 \in LA \cup \Gamma$

Observe that the formula

$$(B_1 \Rightarrow (A \Rightarrow B_1))$$

is a particular case of the axiom **A2** of **H**

By assumption $B_1 \in LA \cup \Gamma$, hence we get the required proof of $(A \Rightarrow B_1)$ from Γ by the following application of the **MP** rule

$$(MP) \frac{B_1 ; (B_1 \Rightarrow (A \Rightarrow B_1))}{(A \Rightarrow B_1)}$$

Deduction Theorem

Case 2 $B_1 = A$

When $B_1 = A$, then to prove

$$\Gamma \vdash (A \Rightarrow B)$$

means to prove $\Gamma \vdash (A \Rightarrow A)$

But $(A \Rightarrow A) \in LA$ (axiom **A1**) of **H**, i.e. $\vdash (A \Rightarrow A)$. By the monotonicity of the consequence we have that

$$\Gamma \vdash (A \Rightarrow A)$$

The above cases **conclude** the proof of the Base Case $i = 1$

Deduction Theorem

Inductive Step

Assume that

$$\Gamma \vdash (A \Rightarrow B_k)$$

for all $k < i$, we will show that using this fact we can conclude that also

$$\Gamma \vdash (A \Rightarrow B_i)$$

Consider a formula B_i in the proof sequence

By the definition, $B_i \in LA \cup \Gamma \cup \{A\}$

or B_i follows by MP from certain B_j, B_m such that $j < m < i$

We have to consider again two cases

Deduction Theorem

Case 1

$$B_i \in LA \cup \Gamma \cup \{A\}$$

The proof of $(A \Rightarrow B_i)$ from Γ in this case is obtained from the proof of the Base Step for $i = 1$ by replacement B_1 by B_i and will be omitted here as a straightforward repetition

Case 2

B_i is a conclusion of MP

If B_i is a conclusion of MP, then we must have two formulas B_j, B_m in the proof sequence, such that $j < i, m < i, j \neq m$ and

$$(MP) \frac{B_j ; B_m}{B_i}$$

By the inductive assumption, the formulas B_j, B_m are such that

$$\Gamma \vdash (A \Rightarrow B_j) \quad \text{and} \quad \Gamma \vdash (A \Rightarrow B_m)$$

Deduction Theorem

Moreover, by the definition of the **Modus Ponens** rule, the formula B_m has to have a form $(B_j \Rightarrow B_i)$, i.e.

$$B_m = (B_j \Rightarrow B_i)$$

and the inductive assumption can be re-written as

$$(*) \quad \Gamma \vdash (A \Rightarrow B_j) \quad \text{and} \quad \Gamma \vdash (A \Rightarrow (B_j \Rightarrow B_i)) \quad \text{for } j < i$$

Observe now that the formula

$$((A \Rightarrow (B_j \Rightarrow B_i)) \Rightarrow ((A \Rightarrow B_j) \Rightarrow (A \Rightarrow B_i)))$$

is a substitution of the axiom **A3** of **H** and hence

$$\vdash ((A \Rightarrow (B_j \Rightarrow B_i)) \Rightarrow ((A \Rightarrow B_j) \Rightarrow (A \Rightarrow B_i)))$$

Deduction Theorem

By the monotonicity,

$$(**) \Gamma \vdash ((A \Rightarrow (B_j \Rightarrow B_i)) \Rightarrow ((A \Rightarrow B_j) \Rightarrow (A \Rightarrow B_i)))$$

Applying the rule **MP** to formulas (*) and (**) i.e. performing the following

$$(MP) \frac{(A \Rightarrow (B_j \Rightarrow B_i)); ((A \Rightarrow (B_j \Rightarrow B_i)) \Rightarrow ((A \Rightarrow B_j) \Rightarrow (A \Rightarrow B_i)))}{((A \Rightarrow B_j) \Rightarrow (A \Rightarrow B_i))}$$

we get that also

$$\Gamma \vdash ((A \Rightarrow B_j) \Rightarrow (A \Rightarrow B_i))$$

Deduction Theorem

Applying again the rule **MP** to formulas (*) and the above

$$\Gamma \vdash ((A \Rightarrow B_j) \Rightarrow (A \Rightarrow B_i))$$

i.e. performing the following

$$(MP) \frac{(A \Rightarrow B_j) ; ((A \Rightarrow B_j) \Rightarrow (A \Rightarrow B_i))}{(A \Rightarrow B_i)}$$

we get that

$$\Gamma \vdash (A \Rightarrow B_i)$$

Deduction Theorem

Finally, suppose that there is some $j < i$ such that

$$B_j \text{ is } \forall x B_j$$

By inductive assumption

$$\Gamma \vdash (A \Rightarrow B_j)$$

and either

(i) B_j **does not** depend upon A or

(ii) x is **not free** variable in A

We want to prove

$$\Gamma \vdash B_j$$

We have these **two** cases (i) and (ii) to consider.

Deduction Theorem

Case (i)

$$\Gamma \vdash (A \Rightarrow B_j)$$

and B_j **does not** depend upon A

Then by **DT Lemma** we have that $\Gamma \vdash B_j$

and, consequently, by the generalization rule (G)

$$\Gamma \vdash \forall x B_j$$

Thus we proved

$$\Gamma \vdash B_j$$

Deduction Theorem

Now, from just proved

$$\Gamma \vdash B_i$$

and axiom **A2** of **H**

$$\vdash (B_i \Rightarrow (A \Rightarrow B_i))$$

and monotonicity

$$\Gamma \vdash (B_i \Rightarrow (A \Rightarrow B_i))$$

and **MP** applied to them we get

$$\Gamma \vdash (A \Rightarrow B_i)$$

Deduction Theorem

Case (ii)

$\Gamma \vdash (A \Rightarrow B_j)$ and x **is not** free variable in A

We know that $\models (\forall x(A \Rightarrow B_j) \Rightarrow (A \Rightarrow \forall xB_j))$

hence the **Completeness Theorem** we get

$\vdash (\forall x(A \Rightarrow B_j) \Rightarrow (A \Rightarrow \forall xB_j))$

Since $\Gamma \vdash (A \Rightarrow B_j)$ by inductive assumption, we get by the generalization rule (**G**) and monotonicity

$$\Gamma \vdash \forall x(A \Rightarrow B_j)$$

By **MP** applied to the above

$$\Gamma \vdash (A \Rightarrow \forall xB_j)$$

That is we got

$$\Gamma \vdash A \Rightarrow B_j$$

Deduction Theorem

Since $\Gamma \vdash (A \Rightarrow B_j)$ by inductive assumption, we get by the generalization rule (G),

$$\Gamma \vdash \forall x(A \Rightarrow B_j)$$

and so, by MP

$$\Gamma \vdash A \Rightarrow \forall xB_j$$

That is we proved

$$\Gamma \vdash (A \Rightarrow B_i)$$

This **completes** the induction and the **proves** part (1) of the **Deduction Theorem**

Deduction Theorem

Deduction Theorem part (2)

The **proof** of the implication

if $\Gamma \vdash (A \Rightarrow B)$ then $\Gamma, A \vdash B$

is straightforward

Assume $\Gamma \vdash (A \Rightarrow B)$. By monotonicity we have also that

$$\Gamma, A \vdash (A \Rightarrow B)$$

Obviously, $\Gamma, A \vdash A$. Applying **MP** to the above, we get the proof of **B** from $\{\Gamma, A\}$ i.e. we have proved that

$$\Gamma, A \vdash B$$

This **ends** the proof of the **Deduction Theorem** for **H**

PART 5: Some other Axiomatizations

Hilbert and Ackermann (1928)

We present here some of **most** known, and historically **important** axiomatizations of classical **predicate** logic, i.e. the following **Hilbert style** proof systems

1. Hilbert and Ackermann (1928)

This formalization is based on **D. Hilbert** and **W. Ackermann** book *Grundzügen der Theoretischen Logik* (Principles of Theoretical Logic), Springer - Verlag, **1928**

The book grew from the **courses** on logic and foundations of mathematics **Hilbert** gave in years **1917-1922**

He received **help** in writeup from **Barnays** and the material was **put into** the book by **Ackermann** and **Hilbert**

Hilbert and Ackermann

The **Hilbert** and **Ackermann** book was conceived as an **introduction** to mathematical logic and was **followed** by another two volumes book written by **D. Hilbert** and **P. Bernays**, *Grundzügen der Mathematik I, II*, Springer-Verlag, **1934, 1939**

Hilbert and **Ackermann** **formulated** and **asked** a question of the **completeness** for their deductive (proof) system

It was **answered** affirmatively by **Kurt Gödel** in **1929** with proof of his **Completeness Theorem**

Hilbert and Ackermann

We define the **Hilbert** and **Ackermann** proof system **HA** following a pattern established for the **H** system

The original **language** used by **Hilbert** and **Ackermann** contained **only** negation \neg and disjunction \cup and so do we

We **define**

$$\mathbf{HA} = (\mathcal{L}_{\{\neg, \cup\}}(\mathbf{P}, \mathbf{F}, \mathbf{C}), \mathcal{F}, \mathbf{LA}, \mathcal{R})$$

where

$$\mathcal{R} = \{(MP), (SB), (G1), (G2)\}$$

The set **LA** of logical axioms is as follows

Hilbert and Ackermann (1928)

Propositional Axioms

$$\text{A1} \quad (\neg(A \cup A) \cup A)$$

$$\text{A2} \quad (\neg A \cup (A \cup B))$$

$$\text{A3} \quad (\neg(A \cup B) \cup (B \cup A))$$

$$\text{A4} \quad (\neg(\neg B \cup C) \cup (\neg(A \cup B) \cup (A \cup C)))$$

for any $A, B, C, \in \mathcal{F}$

Quantifiers Axioms

$$\text{Q1} \quad (\neg\forall xA(x) \cup A(x))$$

$$\text{Q2} \quad (\neg A(x) \cup \exists xA(x))$$

$$\text{Q3} \quad (\neg A(x) \cup \exists xA(x)),$$

for any $A(x) \in \mathcal{F}$

Hilbert and Ackermann

Rules of Inference \mathcal{R}

(MP) is the **Modus Ponens** rule. It has, in the language $\mathcal{L}_{\{\neg, \cup\}}$, a form

$$(MP) \frac{A ; (\neg A \cup B)}{B}$$

(SB) is a **substitution rule**

$$(SB) \frac{A(x_1, x_2, \dots, x_n)}{A(t_1, t_2, \dots, t_n)}$$

where $A(x_1, x_2, \dots, x_n) \in \mathcal{F}$ and $t_1, t_2, \dots, t_n \in \mathbf{T}$

.

Hilbert and Ackermann

(G1), (G2) are **quantifiers generalization rules**

$$(G1) \frac{(\neg B \cup A(x))}{(\neg B \cup \forall x A(x))}$$

$$(G2) \frac{(\neg A(x) \cup B)}{(\neg \exists x A(x) \cup B)}$$

where $A(x), B \in \mathcal{F}$ and B is such that x is **not free** in B

Hilbert and Ackermann

The **HA** system is usually written now with the use of **implication**, i.e. is based on a language

$$\mathcal{L} = \mathcal{L}_{\{\neg, \Rightarrow\}}(\mathbf{P}, \mathbf{F}, \mathbf{C})$$

We define

$$\mathbf{HAI} = (\mathcal{L}_{\{\neg, \Rightarrow\}}(\mathbf{P}, \mathbf{F}, \mathbf{C}), \mathcal{F}, \mathbf{LA}, \mathcal{R})$$

for

$$\mathcal{R} = \{(MP), (SB), (G1), (G2)\}$$

and the set **LA** of logical axioms as follows

Hilbert and Ackermann

Propositional Axioms

$$\mathbf{A1} \quad ((A \cup A) \Rightarrow A)$$

$$\mathbf{A2} \quad (A \Rightarrow (A \cup B))$$

$$\mathbf{A3} \quad ((A \cup B) \Rightarrow (B \cup A))$$

$$\mathbf{A4} \quad ((\neg B \cup C) \Rightarrow ((A \cup B) \Rightarrow (A \cup C)))$$

for any

$$A, B, C, \in \mathcal{F}$$

Quantifiers Axioms

$$\mathbf{Q1} \quad (\forall x A(x) \Rightarrow A(x))$$

$$\mathbf{Q2} \quad (A(x) \Rightarrow \exists x A(x))$$

for any $A(x) \in \mathcal{F}$

Hilbert and Ackermann

Rules of Inference \mathcal{R}

(MP) is Modus Ponens rule

$$(MP) \frac{A ; (A \Rightarrow B)}{B}$$

for any formulas $A, B \in \mathcal{F}$

(SB) is a **substitution rule**

$$(SB) \frac{A(x_1, x_2, \dots, x_n)}{A(t_1, t_2, \dots, t_n)}$$

where $A(x_1, x_2, \dots, x_n) \in \mathcal{F}$ and $t_1, t_2, \dots, t_n \in \mathbf{T}$

Hilbert and Ackermann

(G1), (G2) are **quantifiers generalization rules**.

$$(G1) \frac{(B \Rightarrow A(x))}{(B \Rightarrow \forall x A(x))}$$

$$(G2) \frac{(A(x) \Rightarrow B)}{(\exists x A(x) \Rightarrow B)}$$

where $A(x), B \in \mathcal{F}$ and B is such that x is **not free** in B

The form of the **quantifiers** axioms Q1, Q2, and **quantifiers generalization** rule (G2) is due to **Bernays**

Mendelson (1987)

Here is the **first order** logic proof system as introduced in Elliott Mendelson's book *Introduction to Mathematical Logic* (1987). Hence the name **HM**

HM is a generalization to the **predicate** language of the proof system H_2 for **propositional** logic defined after Mendelson's book and studied in Chapter 5

$$\mathbf{HM} = (\mathcal{L}_{\{\neg, \cup\}}(\mathbf{P}, \mathbf{F}, \mathbf{C}), \mathcal{F}, LA, \mathcal{R} = \{(MP), (G)\})$$

The **HM** components are as follows

Mendelson (1987)

Propositional Axioms

$$A1 \quad (A \Rightarrow (B \Rightarrow A))$$

$$A2 \quad ((A \Rightarrow (B \Rightarrow C)) \Rightarrow ((A \Rightarrow B) \Rightarrow (A \Rightarrow C)))$$

$$A3 \quad ((\neg B \Rightarrow \neg A) \Rightarrow ((\neg B \Rightarrow A) \Rightarrow B))$$

for any $A, B, C, \in \mathcal{F}$

Mendelson

Quantifiers Axioms

$$\text{Q1} \quad (\forall x A(x) \Rightarrow A(t))$$

where t is a term, $A(t)$ is a result of **substitution** of t for all **free** occurrences of x in $A(x)$ and t is **free for x** in $A(x)$, i.e. **no** occurrence of a variable in t becomes a **bound** occurrence in $A(t)$

$$\text{Q2} \quad (\forall x (B \Rightarrow A(x)) \Rightarrow (B \Rightarrow \forall x A(x)))$$

where $A(x), B \in \mathcal{F}$ and B is such that x is **not free** in B

Mendelson

Rules of Inference \mathcal{R}

(MP) is the **Modus Ponens** rule

$$(MP) \frac{A ; (A \Rightarrow B)}{B}$$

for any formulas $A, B \in \mathcal{F}$

(G) is the **generalization rule**

$$(G) \frac{A(x)}{\forall x A(x)}$$

where $A(x) \in \mathcal{F}$ and $x \in VAR$

Rasiowa and Sikorski (1950)

Rasiowa, Sikorski (1950)

Helena Rasiowa and **Roman Sikorski** are the authors of the first **algebraic proof** of the **Gödel completeness theorem** ever given in **1950**

Other **algebraic** proofs were later given by **Rieger, Beth, Łos** in **1951** , and **Scott** in **1954**

Rasiowa and Sikorski (1950)

Here is **Rasiowa- Sikorski** original formalization

$$RS = (\mathcal{L}_{\{\neg, \cap, \cup, \Rightarrow\}}(\mathbf{P}, \mathbf{F}, \mathbf{C}), \mathcal{F}, LA, \mathcal{R})$$

for

$$\mathcal{R} = \{(MP), (SB), (Q1), (Q2), (Q3), (Q4)\}$$

The logical axioms **LA** are as follows

Propositional Axioms

$$\mathbf{A1} \quad ((A \Rightarrow B) \Rightarrow ((B \Rightarrow C) \Rightarrow (A \Rightarrow C)))$$

$$\mathbf{A2} \quad (A \Rightarrow (A \cup B))$$

$$\mathbf{A3} \quad (B \Rightarrow (A \cup B))$$

Rasiowa and Sikorski

$$\mathbf{A4} \quad ((A \Rightarrow C) \Rightarrow ((B \Rightarrow C) \Rightarrow ((A \cup B) \Rightarrow C)))$$

$$\mathbf{A5} \quad ((A \cap B) \Rightarrow A)$$

$$\mathbf{A6} \quad ((A \cap B) \Rightarrow B)$$

$$\mathbf{A7} \quad ((C \Rightarrow A) \Rightarrow ((C \Rightarrow B) \Rightarrow (C \Rightarrow (A \cap B))))$$

$$\mathbf{A8} \quad ((A \Rightarrow (B \Rightarrow C)) \Rightarrow ((A \cap B) \Rightarrow C))$$

$$\mathbf{A9} \quad (((A \cap B) \Rightarrow C) \Rightarrow (A \Rightarrow (B \Rightarrow C)))$$

$$\mathbf{A10} \quad (A \cap \neg A) \Rightarrow B$$

$$\mathbf{A11} \quad ((A \Rightarrow (A \cap \neg A)) \Rightarrow \neg A)$$

$$\mathbf{A12} \quad (A \cup \neg A)$$

for any $A, B, C \in \mathcal{F}$

Rules of Inference \mathcal{R}

(*MP*) is **Modus Ponens** rule

$$(MP) \frac{A ; (A \Rightarrow B)}{B}$$

for any formulas $A, B \in \mathcal{F}$

(*SB*) is a **substitution** rule

$$(SB) \frac{A(x_1, x_2, \dots, x_n)}{A(t_1, t_2, \dots, t_n)}$$

where $A(x_1, x_2, \dots, x_n) \in \mathcal{F}$ and $t_1, t_2, \dots, t_n \in \mathbf{T}$

Rasiowa and Sikorski

(G1), (G2) are the following **quantifiers introduction rules**

$$(G1) \frac{(B \Rightarrow A(x))}{(B \Rightarrow \forall x A(x))}$$

$$(G2) \frac{(A(x) \Rightarrow B)}{(\exists x A(x) \Rightarrow B)}$$

where $A(x), B \in \mathcal{F}$ and B is such that x is **not free** in B

Rasiowa and Sikorski

(G3), (G3) are the following **quantifiers elimination rules**.

$$(G3) \frac{(B \Rightarrow \forall x A(x))}{(B \Rightarrow A(x))}$$

$$(G4) \frac{\exists x(A(x) \Rightarrow B)}{(A(x) \Rightarrow B)}$$

where $A(x), B \in \mathcal{F}$ and B is such that x is **not free** in B

Rasiowa and Sikorski

The **algebraic logic** starts from purely **logical** considerations, **abstracts** from them, places them into a **general algebraic** context, and makes use of **other branches** of mathematics such as **topology**, **set theory**, and **functional analysis**

For **example**, **Rasiowa** and **Sikorski algebraic generalization** of the **completeness theorem** for classical **predicate logic** is the following

Algebraic Completeness Theorem (Rasiowa, Sikorski 1950)

For every formula A of the classical predicate calculus RS the following conditions are **equivalent**

- i A is derivable in RS ;
- ii A is valid in every realization of \mathcal{L} ;
- iii A is valid in every realization of \mathcal{L} in any complete Boolean algebra;
- iv A is valid in every realization of \mathcal{L} in the field $B(X)$ of all subsets of any set $X \neq \emptyset$;

Rasiowa and Sikorski

- v A is valid in every semantic realization of \mathcal{L} in any enumerable set;
- vi there exists a non-degenerate Boolean algebra \mathcal{A} and an infinite set J such that A is valid in every realization of \mathcal{L} in J and \mathcal{A} ;
- vii $A_R(\mathbf{I}) = V$ for the canonical realization R of \mathcal{L} in the Lindenbaum-Tarski algebra \mathcal{LT} of RS and the identity valuation \mathbf{I} ;
- viii A is a predicate tautology.