cse371/math371
LOGIC

Professor Anita Wasilewska

LECTURE 1

LOGICS FOR COMPUTER SCIENCE: CLASSICAL and NON-CLASSICAL

CHAPTER 1
Paradoxes and Puzzles

Chapter 1
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Chapter 1
PART1: Mathematical Paradoxes

Mathematical Paradoxes

Early Intuitive Approach:

Until recently, till the end of the 19th century, mathematical theories used to be built in the intuitive, or axiomatic way.

Historical development of mathematics has shown that it is not sufficient to base theories only on an intuitive understanding of their notions

Example

Consider the following.

By a set, we mean intuitively, any collection of objects.

For example, the set of all even integers or the set of all students in a class.

The objects that make up a set are called its members (elements)

Sets may themselves be members of sets for example, the set of all sets of integers has sets as its members

Example

Sets may themselves be members of sets for example, the set of all sets of integers has sets as its members

Most sets are not members of themselves;

the set of all students, for example, is not a member of itself, because the set of all students is not a student

However, there may be sets that do belong to themselves - for example, the set of all sets

Russell Paradox, 1902

Russell Paradox (1902)

Consider the set A of all those sets X such that X **is not** a member of X

Clearly, A **is** a member of A if and only if A **is not** a member of A

So, if A **is** a member of A, the A is also **not** a member of A; and if A **is not** a member of A, then A **is** a member of A

In any case, A **is** a member of A and A **is not** a member of A

Contradiction

Russell Paradox Solution

Russel proposed and developed a **theory of types** as a solution to the Russel Paradox

The **idea** is that every object must have a definite non-negative integer as its **type** assigned to it

An expression: " x **is a member** of the **set** y"
is meaningful if and only if
the **type** of y is one greater than the **type** of x

Russell Paradox Solution

Russell's **theory of types** guarantees that it is meaningless to say that a **set** belongs to **itself**

Hence Russell's **solution** is:
The set A as stated in the Russell Paradox **does not exist**

The **Type Theory** was extensively developed by Whitehead and Russell in years 1910 - 1913

Logical Paradoxes

Logical Paradoxes, also called Logical Antinomies are **paradoxes** concerning the **notion of a set**

A development of **Axiomatic Set Theory** as one of the most important fields of modern Mathematics, or more specifically of Mathematical Logic or Foundations of Mathematics resulted from the **search for solutions** to various **Logical Paradoxes**

First paradoxes free **Axiomatic Set Theory** was developed by Zermello in 1908

Logical Paradoxes

Two of the most known logical paradoxes (antinomies), other then Russell 's Paradox are those of Cantor and Burali-Forti

They were stated at the end of 19th century

Cantor Paradox involves the theory of cardinal numbers

Burali-Forti Paradox is the analogue to Cantor's but in the theory of ordinal numbers

Cardinality of Sets

We say that sets X and Y have the same **cardinality**, $cardX = cardY$, or that they are **equinumerous** if and only if there is one-to-one correspondence that maps X onto Y

We say that $cardX \leq cardY$ if and only if the set X is **equinumerous** with a **subset** of the set Y

We say that $cardX < cardY$ if and only if $cardX \leq cardY$ and $cardX \neq cardY$

Cantor and Schröder- Berstein Theorems

Cantor Theorem

For any set X,

$cardX < card\mathcal{P}(X)$

Schröder- Berstein Theorem

For any sets X and Y,

If $cardX \leq cardY$ and $cardY \leq cardX$, then

$cardX = cardY$

Cantor Paradox

Cantor Paradox (1899)

Let C be the **universal set** - that is, the set of all sets

Now, $\mathcal{P}(C)$ is a subset of C, so it follows easily that

$card\mathcal{P}(C) \leq cardC$

On the other hand, by **Cantor Theorem**,

$cardC < card\mathcal{P}(C) \leq card\mathcal{P}(C)$, so also $cardC \leq card\mathcal{P}(C)$

From **Schröder- Berstein** theorem we have that
$card\mathcal{P}(C) = cardC$, what **contradicts** **Cantor Theorem**

Solution: **Universal set does not exist.**

Intuitionism

A more **radical interpretation** of the paradoxes has been advocated by Brouwer and his **intuitionist school**

Intuitionists refuse to accept the universality of certain basic logical laws, such as the law of **excluded middle:**
A or not A

For intuitionists the **excluded middle** law is true for **finite sets**, but it is invalid to extend it to **all sets**

The intuitionists' concept of infinite set differs from that of **classical mathematicians**

Chapter 1

PART 2 : Semantic Paradoxes

Semantic Paradoxes

The development of **axiomatic theories** solved some, but not all problems brought up by the **Logical Paradoxes.**

Even the consistent sets of axioms, as the following examples show, do not prevent the occurrence of another kind of paradoxes, called **Semantic Paradoxes**

The **Semantic Paradoxes** deal with the notion of **truth**

Semantic Paradoxes

Berry Paradox, 1906:

Let A denote the set of all positive integers which can be defined in the English language by means of a sentence containing at most 1000 letters

The set A is finite since the set of all sentences containing at most 1000 letters is finite. Hence, there exist positive integer which do not belong to A.

Consider a sentence: n is the least positive integer which cannot be defined by means of a sentence of the English language containing at most 1000 letters

This sentence contains **less than 1000 letters** and defines a positive integer n

Therefore $n \in A$ - but $n \notin A$ by the definition of n

CONTRADICTION!

Berry Paradox Analysis

The paradox resulted entirely from the fact that
we did not say precisely what **notions and sentences** belong to the arithmetic and what **notions and sentences** concern the arithmetic

Of course we didn't talk about and examine arithmetic as a fix and closed deductive system

We also **incorrectly mixed** the natural language with mathematical language of arithmetic

Berry Paradox Solution

We have to distinguish always the **language of the theory** (arithmetic) and the language which **talks about the theory**, called a metalanguage

In general **we must distinguish a formal theory** from the **meta-theory**

In well and correctly defined theory the such paradoxes can not appear

The Liar Paradox

A man says: I am lying.

If he is lying , then what he says is true, and so he is not lying

If he is not lying, then what he says is not true,

and so he is lying

CONTRADICTION!

Liar Paradoxes

These paradoxes arise because the concepts of the type

" I am true", " this sentence is true", " I am lying"

should not occur in the **language** of the theory

They belong to a **metalanguage** of the theory

It it means they belong to a language that talks **about the theory**

Cretan Paradox

The Liar Paradox is a corrected version of a following paradox stated in antiquity by a Cretan philosopher **Epimenides**

Cretan Paradox

The Cretan philosopher Epimenides said: All Cretans are liars

If what he said is **true** , then, since Epimenides is a Cretan, it must be **false**

Hence, what he said is false. Thus, there is a Cretan who is not a liar

CONTRADICTION with what he said: "All Cretans are liars"

Chapter 1
PART 3: Logics for Computer Science

Classical and Intuitionistic

The use of Classical Logic in **computer science** is known, indisputable, and well established.

The existence of PROLOG and Logic Programming as a separate field of computer science is the best example of it.

Intuitionistic Logic in the form of Martin-Löf's theory of types (1982), provides a complete theory of the process of program specification, construction, and verification.

A similar theme has been developed by Constable (1971) and Beeson (1983)

Modal Logics

In 1918, an American philosopher, C.I. Lewis proposed yet another interpretation of lasting consequences, of the logical implication.

In an attempt to avoid, what some felt, the paradoxes of implication (a false sentence implies any sentence) he created a modal logic.

The idea was to distinguish **two sorts of truth**: necessary truth and mere possible (contingent) truth

A possibly true sentence is one which, though true, could be false

Modal Logics for Computer Science

Modal Logics in Computer Science are used as as a tool for analyzing such notions as knowledge, belief, tense.

Modal logics have been also employed in a form of Dynamic logic (Harel 1979) to facilitate the statement and proof of properties of programs

Temporal Logics

Temporal Logics were created for the specification and verification of **concurrent programs** by Harel, Parikh in 1979, 1983 and for a specification of **hardware circuits** by Halpern, Manna, Maszkowski in 1983

They were also used to specify and clarify the concept of causation and its role in **commonsense reasoning** by Shoham in 1988

Fuzzy Sets, Rough Sets, Many valued logics were created and developed to reasoning with **incomplete information.**

Non-classical Logics

The development of new logics and the applications of logics to different areas of Computer Science and in particular to Artificial Intelligence is a subject of a book in itself but is beyond the scope of this book

The course examines in detail the classical logic and some aspects of the intuitionistic logic and its **relationship** with the classical logic

It introduces some of the most standard many valued logics, and examines modal S4, S5 logics.

] It also shows the relationship between the modal S4 and the intuitionistic logics.

Chapter 1
PART 4: Computer Science Puzzles

Reasoning in Artificial Intelligence

Assumption 1:

Flexibility of reasoning is one of the key property of intelligence

Assumption 2:

Commonsense inference is **defeasible** in its nature; we are all capable of drawing conclusions, acting on them, and then **retracting** them if necessary in the face of **new evidence**

Reasoning in Artificial Intelligence

If computer programs are to act **intelligently**, they will need to be similarly **flexible**

Goal:
development of formal systems (logics) that describe commonsense flexibility.

Flexible Reasoning

Example: Reiter, 1987

Consider a statement Birds fly. Tweety, we are told, is a bird.

From this, and the fact that birds fly, we conclude that Tweety can fly

This conclusion is **defeasible:** Tweety may be an ostrich, a penguin, a bird with a broken wing, or a bird whose feet have been set in concrete.

This is a **non-monotonic reasoning:** on learning a new fact (that Tweety has a broken wing), we are forced to **retract** our conclusion (that he could fly)

Non-Monotonic and Default Reasoning

Definition:

A **non-monotonic** reasoning is a reasoning in which the introduction of a new information can **invalidate** old facts

Definition:

A **default** reasoning (logic) is a reasoning that let us draw of plausible inferences from less-than- conclusive evidence in the absence of information to the contrary

Observe: non-monotonic reasoning is an example of default reasoning

Believe Reasoning

Example: Moore, 1983

Consider my reason for believing that I do not have an older brother

It is surely not that one of my parents once casually remarked, You know, you don't have any older brothers, nor have I pieced it together by carefully sifting other evidence

I simply believe that if I did have an older brother I would know about it;

therefore since I don't know of any older brothers of mine, I must not have any

Auto-epistemic Reasoning

The brother example reasoning is not default reasoning nor non-monotonic reasoning

It is a reasoning about one's own **knowledge** or **belief**

Definition

Any reasoning about one's own **knowledge** or **belief** is called an **auto-epistemic** reasoning

Auto-epistemic reasoning **models** the reasoning of an ideally rational agent **reflecting upon** his beliefs or knowledge

Logics which describe it are called **auto-epistemic logics**

Computer Science Puzzles
Missionaries and Cannibals

Example: McCarthy, 1985

Here is the old Cannibals Problem:

Three missionaries and three cannibals come to the river

A rowboat that seats two is available.

If the cannibals ever **outnumber** the missionaries on either bank of the river, the missionaries will be **eaten**

How shall they cross the river?

Traditionally the puzzler is expected to devise a strategy of rowing the boat back and forth that gets them all across and the **disaster.**

Traditional Solution

A state is a triple comprising the number of missionaries, cannibals and boats on the starting bank of the river.

The initial state is 331, the desired state is 000

A solution is given by the sequence:

$$331, 220, 321, 300, 311, 110, 221, 020, 031, 010, 021, 000.$$

Missionaries and Cannibals Revisited

Imagine now giving someone a problem, and after **he puzzles** for a while, he suggests going upstream half a mile and crossing on a bridge

What a bridge? you say.

No bridge is mentioned in the statement of the problem.

He replies: Well, they don't say the isn't a bridge.

So you modify the problem to exclude the bridges and pose it again.

He proposes a helicopter, and after you exclude that, he proposes a winged horse....

Missionaries and Cannibals Revisited

Finally, you tell him the solution.

He attacks your solution on the grounds that the boat might have a leak.

After you rectify that omission from the statement of the problem, he suggests that a sea monster may swim up the river and may swallow the boat

Finally, you must look for a mode of reasoning that will settle his hash once and for all.

McCarthy Solution

McCarthy proposes **circumscription** as a technique for solving his puzzle.

He argues that it is a part of **common knowledge** that a boat can be used to cross the river **unless** there is something with it or something else **prevents** using it

If our facts do not require that there be something that prevents crossing the river, the **circumscription** will generate the conjecture that there isn't

Lifschits has shown in 1987 that in some special cases the **circumscription** is equivalent to a first order sentence.

In those cases we can go back to our secure and well known classical logic

Chapter 1
Paradoxes and Puzzles

PART 4: A Short Review

Definitions and Facts

Definition

Logical Paradoxes, also called Logical Antinomies are paradoxes concerning the **notion of a set**

Definition

Semantic Paradoxes are paradoxes that deal with the notion of **truth**

Definition

A non-monotonic inference is a reasoning in which **introduction** of a new information can **invalidate** old facts

Definitions and Facts

Fact

Non-monotonic reasoning is an example of the default reasoning

Definition

An auto-epistemic reasoning is any reasoning about one's own **knowledge** or **belief**

Auto-epistemic reasoning **models** the reasoning of an ideally rational agent **reflecting** upon his beliefs or knowledge

Definitions and Facts

Facts

The main difference between **classical** and **intuitionists'** mathematics lies in the **interpretation** of the word **exists**

In classical mathematics proving **existence** of an object x such that a property $P(x)$ holds **does not** always mean that one is able to indicate a method of its construction

In the **intuitionists' universe** we are justified in asserting the **existence** of an object having a certain property only if we know an **effective method** for constructing, or finding such an object