CSE371 QUIZ 2 SOLUTIONS Spring 2020
(25pts + 15pts extra credit)

PROBLEM 1 (15pts)

S is the following proof system:

$$S = (\mathcal{L}_{\{\Rightarrow,\cup,\neg\}},\ \mathcal{F},\ LA = \{(A \Rightarrow (A \cup B))\}\ (r1),\ (r2))$$

Rules of inference:

$$(r1)\ \frac{A\ ;\ B}{(A \cup \neg B)}, \qquad (r2)\ \frac{A\ ;\ (A \cup B)}{B}$$

1. (5pt) Verify whether S is sound/not sound under **classical** semantics.

Solution

(1pt) The Logical Axiom is a BASIC TAUTOLOGY (you can also prove it by contradiction)

(1pt) The rule $(r1)$ is sound because for any v, if $v^*(A) = T$,

then $v^*((A \cup \neg B)) = v^*(A) \cup v^*(B) = T \cup v^*(B) = T$, for all formulas B.

(3pt) The rule $(r1)$ is **not sound**

Take any v such that it evaluates $A = T$ and $B = F$. The premiss $(A \cup B$ of the rule $(r2)$ is T and the conclusion B is F.

The system is **not sound**

2. (5pts) Find a **formal proof** in S of the formula $\neg(A \Rightarrow (A \cup B))$, i. e. show that

$$\vdash_S \neg(A \Rightarrow (A \cup B))$$

Solution The proof is as follows

B_1**:** $(A \Rightarrow (A \cup B))$, (axiom)

B_2**:** $(A \Rightarrow (A \cup B))$, (axiom)

B_3**:** $((A \Rightarrow (A \cup B)) \cup \neg(A \Rightarrow (A \cup B)))$, (rule (r1) application to B_1 and B_2)

B_4**:** $\neg(A \Rightarrow (A \cup B))$, (rule (r2) application to B_1 and B_3).

3. (5pt) Does above point **2.** prove that $\models \neg(A \Rightarrow (A \cup B))$?

Solution

(3pts) We proved that the proof system S is **not sound**, so existence of a proof does not guarantee that what we proved is a tautology.

(2pts) Moreover, the proof of $\neg(A \Rightarrow (A \cup B))$ used rule $(r2)$ that is not sound!

PROBLEM 2 (15pts)

Let S be the following **proof system** $S = (\mathcal{L}_{\{\neg,\mathbf{L},\cup,\Rightarrow\}}, \mathcal{F}, \{\mathbf{A1},\mathbf{A2}\}, \{r1, r2\})$

for the logical axioms and rules of inference defined for any formulas $A, B \in \mathcal{F}$ as follows

Logical Axioms

A1 $(\mathbf{L}A \cup \neg\mathbf{L}A)$, **A2** $(A \Rightarrow \mathbf{L}A)$

Rules of inference:

$$(r1)\ \frac{A\ ;B}{(A \cup B)}, \qquad (r2)\ \frac{A}{\mathbf{L}(A \Rightarrow B)}$$

1. (8pts) Show, by constructing a proper **formal proof** that

$$\vdash_S ((\mathbf{L}b \cup \neg\mathbf{L}b) \cup \mathbf{L}((\mathbf{L}a \cup \neg\mathbf{L}a) \Rightarrow b)))$$

Write all steps of the **formal proof** with **comments** how each step was obtained.

Solution

Here is the proof B_1, B_2, B_3, B_4

B_1: $(\mathbf{L}a \cup \neg\mathbf{L}a)$ Axiom A_1 for A= a

B_2: $\mathbf{L}((\mathbf{L}a \cup \neg\mathbf{L}a) \Rightarrow b)$ rule r2 for B=b applied to B_1

B_3: $(\mathbf{L}b \cup \neg\mathbf{L}Ab)$ Axiom A_1 for A=b

B_4: $((\mathbf{L}b \cup \neg\mathbf{L}b) \cup \mathbf{L}((\mathbf{L}a \cup \neg\mathbf{L}a) \Rightarrow b))$ r1 applied to B_3 and B_2

2. (7pts) Does the above point **1.** PROVE that $\models_{\mathbf{M}} ((\mathbf{L}b \cup \neg\mathbf{L}b) \cup \mathbf{L}((\mathbf{L}a \cup \neg\mathbf{L}a) \Rightarrow b)))$? for the semantics **M** defined below JUSTIFY, in detail your answer.

Solution

(1pts) Point **1.**, i.e. the fact that $\vdash_S ((\mathbf{L}b \cup \neg\mathbf{L}b) \cup \mathbf{L}((\mathbf{L}a \cup \neg\mathbf{L}a) \Rightarrow b)))$ **PROVES** that

$\models_{\mathbf{M}} ((\mathbf{L}b \cup \neg\mathbf{L}b) \cup \mathbf{L}((\mathbf{L}a \cup \neg\mathbf{L}a) \Rightarrow b)))$ **ONLY WHEN** the proof system S **is sound**'

HENCE we have to VERIFY the **soundness** of the system S

WE do it in the following STEPS.

s1 We VERIFY if the both logical axioms of S are **M tautologies**

(1pt) We verify that **A1** is **M** tautology:

$\mathbf{L}T \cup \neg\mathbf{L}T = T \cup F = T$, $\mathbf{L} \perp \cup \neg\mathbf{L} \perp = F \cup \neg F = F \cup T = T$, $\mathbf{L}F \cup \neg\mathbf{L}F = F \cup \neg F = T$

(1pt) We verify that **A2** is **M** tautology in a similar way:

$(T \Rightarrow \mathbf{L}T) = T$, $(F \Rightarrow \mathbf{L}F) = T$, and $(\perp \Rightarrow \mathbf{L} \perp) = T$

s2 We VERIFY if the both rules of inference **are sound**

(1pt) Rule r1 is **sound** because when $A = T$ and $B = T$ we get $A \cup B = T \cup T = T$

(1pts) Rule r2 is **not sound** because when $A = T$ and $B = F$ (or $B = \perp$) we get $\mathbf{L}(A \Rightarrow B) = \mathbf{L}(T \Rightarrow F) = \mathbf{L}F = F$ or $\mathbf{L}(T \Rightarrow \perp) = \mathbf{L} \perp = F$

(2pts) ANSWER: **No, it doesn't** because the system S is **not sound**

We define a 3 valued extensional semantics **M** for the language $\mathcal{L}_{\{\neg, \mathbf{L}, \cup, \Rightarrow\}}$ by **defining the connectives** $\neg, \cup, \Rightarrow$ on a set $\{F, \perp, T\}$ of logical values as follows.

L Connective

L	F	⊥	T
	F	*F*	T

Negation :

¬	F	⊥	T
	T	*F*	F

Implication

⇒	F	⊥	T
F	T	T	T
⊥	*T*	⊥	T
T	F	*F*	T

Disjunction :

∪	F	⊥	T
F	F	⊥	T
⊥	⊥	T	T
T	T	*T*	T

PROBLEM 3 (10pts)

Consider the Hilbert system $H1 = (\mathcal{L}_{\{\Rightarrow\}}, \ \mathcal{F}, \ \{A1, A2\}, \ (MP) \frac{A \ ; \ (A \Rightarrow B)}{B})$ where for any $A, B \in \mathcal{F}$

$A1; \ (A \Rightarrow (B \Rightarrow A)), \quad A2: \ ((A \Rightarrow (B \Rightarrow C)) \Rightarrow ((A \Rightarrow B) \Rightarrow (A \Rightarrow C)))$.

1. (5pts) The **Deduction Theorem** holds for $H1$. Use the **Deduction Theorem** to show that $(A \Rightarrow (C \Rightarrow B)) \vdash_{H1} (C \Rightarrow (A \Rightarrow B))$

Remark: This is not the only solution; it is just the easiest. IF you apply the Deduction Theorem ONCE you get another solution.

Solution

We apply the **Deduction Theorem** twice, i.e. we get

$(A \Rightarrow (C \Rightarrow B)) \vdash_H (C \Rightarrow (A \Rightarrow B))$ if and only if

$(A \Rightarrow (C \Rightarrow B)), \ C \vdash_H (A \Rightarrow B)$ if and only if

$(A \Rightarrow (C \Rightarrow B)), \ C, \ A \vdash_H B$

We now construct a proof of $(A \Rightarrow (C \Rightarrow B)), C, A \vdash_H B$ as follows

$B_1: \quad (A \Rightarrow (C \Rightarrow B))$ hypothesis

$B_2: \quad C$ hypothesis

$B_3: \quad A$ hypothesis

$B_4: \quad (C \Rightarrow B) \quad B_1, B_3$ and (MP)

B_5 : B B_2 , B_4 and (MP)

3. (5pts) Let $H2$ be the proof system obtained from the system $H1$ by **extending the language** to contain the negation ¬ and **adding** one additional axiom:

A3 $((\neg B \Rightarrow \neg A) \Rightarrow ((\neg B \Rightarrow A) \Rightarrow B)))$.

Explain why the **Deduction Theorem** holds for H2.

Solution

The proof of the Deduction Theorem for H1 used only axioms **A1, A2**

Adding axiom **A3** (and adding ¬ to the language) does not change anything in the proof.

Hence **Deduction Theorem** holds for H2