PROBLEM 1 (15pts)

$S$ is the following proof system:

$S = (L[⇒,∪,¬], T, \ LA = \{(A \Rightarrow (A \cup B))\} \ (r1), \ (r2) )$

Rules of inference:

$\frac{A ; \ B}{(A \cup \neg B)} \quad \frac{A ; (A \cup B)}{B}$

1. (5pt) Verify whether $S$ is sound/not sound under **classical** semantics.

Solution

(1pt) The Logical Axiom is a **BASIC TAUTOLOGY** (you can also prove it by contradiction)

(1pt) The rule ($r1$) is sound because for any $v$, if $v^*(A) = T$,

then $v^*((A \cup \neg B)) = v^*(A) \cup v^*(B) = T \cup v^*(B) = T$, for all formulas $B$.

(3pt) The rule ($r1$) is **not sound**

Take any $v$ such that it evaluates $A = T$ and $B = F$. The premiss $(A \cup B)$ of the rule $(r2)$ is $T$ and the conclusion $B$ is $F$. The system is **not sound**

2. (5pts) Find a formal proof in $S$ of the formula $\neg(A \Rightarrow (A \cup B))$, i. e. show that

$\vdash_S \neg(A \Rightarrow (A \cup B))$

Solution

The proof is as follows

$B_1$: $(A \Rightarrow (A \cup B))$, (axiom)

$B_2$: $(A \Rightarrow (A \cup B))$, (axiom)

$B_3$: $((A \Rightarrow (A \cup B)) \cup \neg (A \Rightarrow (A \cup B)))$, (rule ($r1$) application to $B_1$ and $B_2$)

$B_4$: $\neg (A \Rightarrow (A \cup B))$, (rule ($r2$) application to $B_1$ and $B_3$).

3. (5pt) Does above point 2. prove that $\models \neg (A \Rightarrow (A \cup B))$?

Solution

(3pts) We proved that the proof system $S$ is **not sound**, so existence of a proof does not guarantee that what we proved is a tautology.

(2pts) Moreover, the proof of $\neg (A \Rightarrow (A \cup B))$ used rule ($r2$) that is not sound!
PROBLEM 2 (15pts)

Let $S$ be the following proof system $S = (\mathcal{L}, \{\neg, \lor, \Rightarrow\}, \mathcal{F}, \{A1, A2\}, \{r1, r2\})$
for the logical axioms and rules of inference defined for any formulas $A, B \in \mathcal{F}$ as follows

**Logical Axioms**

$A1$ ($\mathcal{L}A \cup \neg \mathcal{L}A$), $A2$ ($A \Rightarrow \mathcal{L}A$)

**Rules** of inference:

\[
\frac{A \land B}{(r1) A \land B}, \quad \frac{A}{(r2) \mathcal{L}A \Rightarrow B}
\]

1. (8pts) Show, by constructing a proper formal proof that

\[\vdash_S ((\mathcal{L}b \cup \neg \mathcal{L}b) \cup \mathcal{L}((\mathcal{L}a \cup \neg \mathcal{L}a) \Rightarrow b))\]

Write all steps of the formal proof with comments how each step was obtained.

**Solution**

Here is the proof $B_1, B_2, B_3, B_4$

$B_1$: ($\mathcal{L}a \cup \neg \mathcal{L}a$) Axiom $A_1$ for $A=a$

$B_2$: $\mathcal{L}((\mathcal{L}a \cup \neg \mathcal{L}a) \Rightarrow b)$ rule $r2$ for $B=b$ applied to $B_1$

$B_3$: ($\mathcal{L}b \cup \neg \mathcal{L}ab$) Axiom $A_1$ for $A=b$

$B_4$: ($\mathcal{L}(\mathcal{L}b \cup \neg \mathcal{L}b) \cup \mathcal{L}((\mathcal{L}a \cup \neg \mathcal{L}a) \Rightarrow b))$ $r1$ applied to $B_3$ and $B_2$

2. (7pts) Does the above point 1. PROVE that $\models_M ((\mathcal{L}b \cup \neg \mathcal{L}b) \cup \mathcal{L}((\mathcal{L}a \cup \neg \mathcal{L}a) \Rightarrow b))$? for the semantics $M$ defined below JUSTIFY, in detail your answer.

**Solution**

(1pts) Point 1., i.e. the fact that $\vdash_S ((\mathcal{L}b \cup \neg \mathcal{L}b) \cup \mathcal{L}((\mathcal{L}a \cup \neg \mathcal{L}a) \Rightarrow b))$ PROVES that

$\models_M ((\mathcal{L}b \cup \neg \mathcal{L}b) \cup \mathcal{L}((\mathcal{L}a \cup \neg \mathcal{L}a) \Rightarrow b))$ ONLY WHEN the proof system $S$ is sound'

HENCE we have to VERIFY the soundness of the system $S$

WE do it in the following STEPS.

s1 We VERIFY if the both logical axioms of $S$ are $M$ tautologies

(1pt) We verify that $A1$ is $M$ tautology:

$\mathcal{L}T \cup \neg \mathcal{L}T = T \cup F = T$, $\mathcal{L} \bot \cup \neg \mathcal{L} \bot = F \cup \neg F = F \cup T = T$, $\mathcal{L}F \cup \neg \mathcal{L}F = F \cup \neg F = T$

(1pt) We verify that $A2$ is $M$ tautology in a similar way:

$(T \Rightarrow \mathcal{L}T) = T$, $(F \Rightarrow \mathcal{L}F) = T$, and $(\bot \Rightarrow \mathcal{L} \bot) = T$

s2 We VERIFY if the both rules of inference are sound
(1pt) Rule r1 is sound because when \( A = T \) and \( B = T \) we get \( A \cup B = T \cup T = T \).

(1pts) Rule r2 is not sound because when \( A = T \) and \( B = F \) (or \( B = \bot \)) we get \( L(A \Rightarrow B) = L(T \Rightarrow F) = LF = F \) or \( L(T \Rightarrow \bot) = L \bot = F \).

(2pts) ANSWER: No, it doesn’t because the system \( S \) is not sound.

We define a 3 valued extensional semantics \( M \) for the language \( \mathcal{L}_{\{\neg, \lor, \Rightarrow\}} \) by defining the connectives \( \neg, \lor, \Rightarrow \) on a set \( \{F, \bot, T\} \) of logical values as follows.

<table>
<thead>
<tr>
<th>( \mathbf{L} ) Connective</th>
<th>Negation</th>
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<tbody>
<tr>
<td>( \begin{array}{ccc} L &amp; F &amp; \bot &amp; T \ \end{array} )</td>
<td>( \begin{array}{ccc} \neg &amp; F &amp; \bot &amp; T \ \end{array} )</td>
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<table>
<thead>
<tr>
<th>Implication</th>
<th>Disjunction</th>
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<tbody>
<tr>
<td>( \Rightarrow )</td>
<td>( \cup )</td>
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<tr>
<td>( \begin{array}{ccc} \Rightarrow &amp; F &amp; \bot &amp; T \ \end{array} )</td>
<td>( \begin{array}{ccc} \cup &amp; F &amp; \bot &amp; T \ \end{array} )</td>
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PROBLEM 3 (10pts)

Consider the Hilbert system \( H1 = (\mathcal{L}_{\{\Rightarrow\}}, \mathcal{F}, \{A1, A2\}, (MP) \dfrac{A : (A \Rightarrow B)}{B} \) where for any \( A, B \in \mathcal{F} \)

\( A1: (A \Rightarrow (B \Rightarrow A)), \ A2: ((A \Rightarrow (B \Rightarrow C)) \Rightarrow ((A \Rightarrow B) \Rightarrow (A \Rightarrow C))) \).

1. (5pts) The Deduction Theorem holds for \( H1 \). Use the Deduction Theorem to show that \( (A \Rightarrow (C \Rightarrow B)) \vdash_{H1} (C \Rightarrow (A \Rightarrow B)) \).

Remark: This is not the only solution; it is just the easiest. IF you apply the Deduction Theorem ONCE you get another solution.

Solution

We apply the Deduction Theorem twice, i.e. we get

\( (A \Rightarrow (C \Rightarrow B)) \vdash_H (C \Rightarrow (A \Rightarrow B)) \) if and only if

\( (A \Rightarrow (C \Rightarrow B)), \ C \vdash_H (A \Rightarrow B) \) if and only if

\( (A \Rightarrow (C \Rightarrow B)), \ C, A \vdash_H B \)

We now construct a proof of \( (A \Rightarrow (C \Rightarrow B)), C, A \vdash_H B \) as follows

\( B_1: (A \Rightarrow (C \Rightarrow B)) \) hypothesis

\( B_2: C \) hypothesis

\( B_3: A \) hypothesis

\( B_4: (C \Rightarrow B) \) \( B_1, B_3 \) and (MP)
3. (5pts) Let $H_2$ be the proof system obtained from the system $H_1$ by extending the language to contain the negation $\neg$ and adding one additional axiom:

$A_3 \ ((\neg B \Rightarrow \neg A) \Rightarrow ((\neg B \Rightarrow A) \Rightarrow B)))$.

Explain why the Deduction Theorem holds for $H_2$.

Solution

The proof of the Deduction Theorem for $H_1$ used only axioms $A_1, A_2$

Adding axiom $A_3$ (and adding $\neg$ to the language) does not change anything in the proof.

Hence Deduction Theorem holds for $H_2$