CSE/MAT371 Q1 SOLUTIONS Spring 2020

QUESTION 1 Write the following natural language statement:

From the fact that each natural number is greater than zero we deduce that it is not possible that $2+2=5$ *or, if it is possible that it is not true that* $2+2=5$*, then it is necessary that it is not true that each natural number is greater than zero*

in the following two ways.

1. As a formula $A_1 \in \mathcal{F}_1$ of a language $\mathcal{L}_{\{\neg,\ \Box,\ \Diamond,\ \cap,\ \cup,\ \Rightarrow\}}$

Solution

Propositional Variables are: a, b

a denotes statement: *each natural number is greater than zero* ,

b denotes a statement: $2+2=5$

Formula $A_1 \in \mathcal{F}_1$ is:

$$(a \Rightarrow (\neg\Diamond b \cup (\Diamond\neg b \Rightarrow \Box\neg a)))$$

2. As a formula $A_2 \in \mathcal{F}_2$ of a language $\mathcal{L}_{\{\neg,\ \cap,\ \cup,\ \Rightarrow\}}$

Solution

Propositional Variables are: a, b, c, d

a denotes statement: *each natural number is greater than zero* ,

b denotes a statement: *it is possible that* $2+2=5$,

c denotes a statement: *it is possible that it is not true that* $2+2=5$,

d denotes a statement: *it is necessary that it is not true that each natural number is greater than zero*

Formula $A_2 \in \mathcal{F}_1$ is

$$(a \Rightarrow (\neg b \cup (c \Rightarrow d)))$$

QUESTION 2

Circle formulas that **are** propositional/ predicate **tautologies**

1. Circle formulas that **are** propositional **tautologies**

$$\mathcal{S}_1 = \{\ (A \Rightarrow (A \cup \neg B)),\quad ((a \Rightarrow b) \cup (a \cap \neg b)),\quad (A \cup (A \Rightarrow B)),\quad (a \cup \neg b)\ \}$$

Solution Obviously, $\not\models (a \cup \neg b)$, all other formulas are tautologies

2. Circle formulas that **are** predicate **tautologies**

$$\mathcal{S}_2 = \{\ (\forall x\, A(x) \Rightarrow \exists x\, A(x)),\quad (\forall x\, (\neg P(x,y) \cup P(x,y)) \Rightarrow \exists x\, P(x,y)),\quad ((\exists x A(x) \cap \exists x B(x)) \Rightarrow \exists x\, (A(x) \cap B(x)))\ \}$$

Solution Obviously, $\models (\forall x\, A(x) \Rightarrow \exists x\, A(x))$, all other formulas are NOT tautologies

QUESTION 3

Here is a mathematical statement **S**:

For all rational numbers $x \in Q$ the following holds: If $x = 0$, then there is a natural number $n \in N$, such that $x+n = 0$

1. Re-write **S** as a symbolic mathematical statement **SM** that only uses mathematical and logical symbols.

Solution **S** becomes a symbolic mathematical statement

$$\mathbf{SM}: \quad \forall_{x \in Q}(x = 0 \Rightarrow \exists_{n \in N}\, x + n = 0)$$

2. Translate the symbolic statement **SM** into to a corresponding formula with **restricted quantifiers**. Explain your choice of symbols.

Solution We write Q(x) for $x \in Q$, N(y) for $y \in N$, a constant c for the number 0. We use $E \in \mathbf{P}$ to denote the relation =, we use $f \in \mathbf{F}$ to denote the function +.

The statement $x = 0$ becomes an **atomic formula** E(x, c). The statement $x + n = 0$ becomes an **atomic formula** E(f(x,y), c).

The symbolic mathematical statement **SM** becomes a **restricted quantifiers** formula

$$\forall_{Q(x)}(E(x, c) \Rightarrow \exists_{N(y)} E(f(x, y), c))$$

3. Translate your **restricted domain** quantifiers logical formula into a correct formula A of the predicate language $\mathcal{L}$

Solution We apply now the **transformation rules** and get a corresponding formula $A \in \mathcal{F}$:

$$\forall x(Q(x) \Rightarrow (E(x, c) \Rightarrow \exists y(N(y) \cap E(f(x, y), c))))$$

QUESTION 4

Given a formula A : $\forall x \exists y\, P(f(x, y), c)$ of the predicate language $\mathcal{L}$, and two **model structures**

$$\mathbf{M_1} = (Z, I_1), \quad \text{and} \quad \mathbf{M_2} = (N, I_2)$$

with the **interpretations** defined as follows.

$$P_{I_1} : =, \quad f_{I_1} : +, \quad c_{I_1} : 0 \quad \text{and} \quad P_{I_2} : >, \quad f_{I_2} : \cdot, \quad c_{I_2} : 0$$

1. Show that $\mathbf{M_1} \models A$

Solution

$\mathbf{M_1} \models A$ because $A_{I_1} : \forall_{x \in Z} \exists_{y \in Z}\, x + y = 0$ is a **true** mathematical statement as we have that each $x \in Z$ exists $y = -x$ and $-x \in Z$ and $x - x = 0$

2. Show that $\mathbf{M_2} \not\models A$

Solution

$\mathbf{M_2} \not\models A$ because $A_{I_2} : \forall_{x \in N} \exists_{y \in N}\, x \cdot y > 0$ is a **false** statement for $x = 0$.

QUESTION 5

Prove that the following formulas $A_1,\ A_2 \in \mathcal{F}$ are **classical tautologies**.

You **can't use** the Truth Tables Method.

Formula $A_1 \in \mathcal{F}$ is

$$((((a \Rightarrow b) \cap \neg c) \Rightarrow ((((a \Rightarrow b) \cap \neg c) \cup \neg d))$$

Formula $A_2 \in \mathcal{F}$ is

$$(((a \Rightarrow b) \cap \neg c) \cup d) \cap \neg e) \Rightarrow (((a \Rightarrow b) \cap \neg c) \cup d) \cap \neg e) \cup ((a \Rightarrow \neg e)))$$

Solution

Both formulas are tautologies because they are particular cases of the basic **propositional tautology** for any $A, B \in \mathcal{F}$

$$(A \Rightarrow (A \cup B))$$

Extra information

If you didn't remember that $\models (A \Rightarrow (A \cup B))$ you could PROVE it by the Proof by Contradiction Method.as follows.

Assume that $\not\models (A \Rightarrow (A \cup B))$

We get that $(A \Rightarrow (A \cup B)) = F$ **iff** $A = T$ and $(A \cup B) = F$

From above we have that $(T \cup B) = F$

This is **impossible** by the definition of $\cup$. We got a **contradiction**, hence

$$\models \quad (A \Rightarrow (A \cup B))$$

Remember that as I wrote in my Lectures and told in class- at this stage you can't use the TTables evaluations. IT will result in **0pts** for the problem.