CSE371 FINAL SOLUTIONS Spring 2020
(75 pts + 15 pts extra credit)

PROBLEM 1 (20pts)

1. (5pts)

Give a definition and an example a Logical Paradox. Explain why it is paradox.

Solution

Logical Paradoxes, also called **Logical Antinomies** are paradoxes concerning **the notion of a set**

Here are three of them

Russel Paradox, 1902, Cantor Paradox, 1899, Burali-Forti Paradox, 1897

Example: Russel paradox.

Consider the set A of all those sets X such that X is not a member of X. Clearly, by definition, A is a member of A if and only if A is not a member of A. So, if A is a member of A, the A is also not a member of A; and if A is not a member of A, then A is a member of A. In any case, A is a member of A and A is not a

2. (5pts)

Give a definition and an example a Semantic Paradox. Explain why it is paradox.

Solution

Semantical paradoxes (antinomies) deal with the notion of truth, or provability. They are caused by a collision between the theory and meta-theory, that is, by inclusion of meta-theoretical statements in the theory.

Example: The Liar Paradox.

A man says: I am lying. *If he is lying, then what he says is true, and so he is not lying. If he is not lying, then what he says is not true, and so he is lying. In any case, he is lying and he is not lying.*

3. (5pts)

Give a definition an example of a default reasoning. Explain why it is a correct example.

Solution

Default reasoning is a reasoning in which it is allowed to draw plausible inferences from less-then- conclusive evidence in the absence of information to the contrary.

Example: Consider a statement *Birds fly*. Tweety, we are told, is a bird. From this, and the fact that birds fly, we conclude that Tweety can fly.

Explanation This conclusion, however is *defeasible*: Tweety may be an ostrich, a penguin, a bird with a broken wing, or a bird whose feet have been set in concrete. But as long as we don't have the evidence to the contrary (*Tweedy has a broken wing*) we accept the conclusion that *Tweety can fly*.

4. (5pts)

Describe a main difference between classical and intuitionists' mathematics.

Solution

One of the **main differences** between classical and intuitionists' mathematics lies in the interpretation of the word **exists**.

PROBLEM 2 (20pts)

Write the following natural language statement:

From the fact that each natural number is equal zero we **deduce** *that: it is* **not possible** *that Anne is a boy* **or**, **if** *it is* **possible** *that Anne is* **not** *a boy,* **then** *it is* **necessary** *that it is* **not** *true that each natural number is equal zero*

in the following two ways

1. (10pts)

As a formula $A_1 \in \mathcal{F}_1$ of a language $\mathcal{L}_{\{\neg, \Box, \Diamond, \cap, \cup, \Rightarrow\}}$

2. As a formula $A_2 \in \mathcal{F}_2$ of a language $\mathcal{L}_{\{\neg, \cap, \cup, \Rightarrow\}}$

Write a **detailed solutions**, explaining all steps used.

1. Solution

For the formula $A_1 \in \mathcal{F}_1$ of a language $\mathcal{L}_{\{\neg, \Box, \Diamond, \cap, \cup, \Rightarrow\}}$

Propositional Variables: a, b , where

a denotes statement: *each natural number is equal zero*,

b denotes statement: *Sun is swimming*

Propositional Modal Connectives: $\Box$, $\Diamond$

$\Diamond$ denotes statement: **it is possible that**, $\Box$ denotes statement: **it is necessary that**

The formula A_1 is

$$(a \Rightarrow (\neg\Diamond b \cup (\Diamond\neg b \Rightarrow \Box\neg a)))$$

2. (10pts)

Solution

For the formula $A_2 \in \mathcal{F}_2$ of a language $\mathcal{L}_{\{\neg, \cap, \cup, \Rightarrow\}}$

Propositional Variables: a, b , c, d where

a denotes statement: *each natural number is equal* zero,

b denotes statement: *possible that Sun is swimming*

c denotes statement: *possible that Sun is not swimming,*

d denotes statement: *necessary that it is not true that each natural number is equal zero*

Formula A_2 is

$$(a \Rightarrow (\neg b \cup (c \Rightarrow d)))$$

PROBLEM 3 (15pts)

1. Given a predicate language $\mathcal{L}(\mathbf{P}, \mathbf{F}, \mathbf{C})$ and a structure $\mathbf{M} = [U, I]$ such that

$U = N$ and $P_I :=$, $g_I : \cdot$, $a_I : 0$, where N is the set of natural numbers.

For the following formula A

$$\forall x \exists y (P(g(x, y), a) \Rightarrow (P(x, a) \cup P(y, a)))$$

decide whether $\mathbf{M} \models A$ or not.

Do so by examining the corresponding mathematical statement defined by $\mathbf{M}$.

Solution

The corresponding mathematical statement defined by $\mathbf{M}$ (written with logical symbols) is

$$\forall n \exists m \, (nm = 0 \ \Rightarrow \ (n = 0 \cup m = 0))$$

It is a TRUE statement in the set N of natural numbers

The mathematical statement can also be written as

$$\forall_{n \in N} \exists_{m \in N} \, (nm = 0 \ \Rightarrow \ (n = 0 \cup m = 0))$$

PROBLEM 4 (15pts)

Let S_3 be a 3-valued extentsional semantics for $\mathcal{L}_{\{\neg, \cap, \cup, \Rightarrow\}}$ defined as follows.

$V_3 = \{F, \perp, T\}$, for $F \leq \perp \leq T$

For **any** $x, y \in \{F, \perp, T\}$ we put

$F \cup F = F$, $F \cup T = T$, $T \cup F = T \cup T = T$, and $x \cup y = \perp$ otherwise

$\neg F = T$, $\neg \perp = F$, $\neg T = \perp$, and $x \cap y = min\{x, y\}$, $x \Rightarrow y = \neg x \cup y$

Consider the following classical tautologies:

$$A_1 = ((\neg a \cup b) \cup \neg(a \Rightarrow b)), \qquad A_2 = (a \Rightarrow (b \Rightarrow a))$$

1. (8pts)

Find S_3 counter-models for A_1, A_2, if exist. Use shorthand notation.

Solution

Here are S_3 Connectives Tables for **Disjunction** and **Negation**

$\cup$	F	$\perp$	T
F	F	$\perp$	T
$\perp$	$\perp$	$\perp$	$\perp$
T	T	$\perp$	T

$\neg$	F	$\perp$	T
	T	F	$\perp$

Reminder: $x \cap y = min\{x, y\}$, $x \Rightarrow y = \neg x \cup y$, for any $x, y \in \{F, \perp, T\}$

Any v such that $v(a) = v(b) =\perp$ is a counter-model for both, A_1 and A_2. couner-model

We evaluate A_1

$$v^*(A_1) = (\neg \perp \cup \perp) \cup \neg(\perp\Rightarrow\perp) =\perp \cup\neg(\perp\Rightarrow\perp) =\perp$$

This is not only counter-model. For example, any v such that $v(a) = T$ is a counter model for A_1 and any v such that $v(b) = T$ is a counter model for A_2.

We evaluate A_2

$$v^*(A_2) =\perp\Rightarrow (\perp\Rightarrow\perp) = \neg \perp \cup(\perp\Rightarrow\perp) = F \cup (\neg \perp \cup \perp) = F\cup \perp=\perp$$

2. (7pts)

Define a 2-valued extensional semantics S_2 with $V_2 = \{F, T\}$, for $F \leq\perp\leq T$ for the language $\mathcal{L}_{\{\neg,\cap,\cup,\Rightarrow\}}$, such that **none of** A_1, A_2 is a S_2 **tautology**. Justify your results. Use shorthand notation.

Solution This is not the only solution, but it is the simplest and most obvious. Here it is.

We define S_2 connectives as follows.

$$\neg x = F, \quad x \Rightarrow y = x \cup y = F \quad \text{for all} \quad x, \ y \in \{F, T\}$$

and $x \cap y$ can be defined in the same, or in any other way, as this connective do not appear in our formulas.

PROBLEM 5 (10pts)

Let $S = (\mathcal{L} = \mathcal{L}_{\{\neg,\cap,\cup,\Rightarrow\}}, \ \mathcal{F}, \ \{A1, A2, A3\}, \ MP \frac{A \ ;(A\Rightarrow B)}{B})$ be a proof system with logical axioms:

A1: $(A \Rightarrow (B \Rightarrow A))$,

A2: $((A \Rightarrow (B \Rightarrow C)) \Rightarrow ((A \Rightarrow B) \Rightarrow (A \Rightarrow C)))$, A3: $((\neg B \Rightarrow \neg A) \Rightarrow ((\neg B \Rightarrow A) \Rightarrow B))$.

1. (5pts)

Explain why S is sound under classical semantics.

Solution 1

Axioms $A1 - A4$ are basic classical **tautologies** and Modus Ponens is a **sound** rule of inference

Solution 2

The proof system S is Chapter 5 Hilbert proof system H_2 that we proved it to be **sound** and **complete**.

2. (5pts)

Show, by constructing a formal proof, that $\vdash_S ((a \Rightarrow b) \Rightarrow (a \Rightarrow b))$

Solution 1

We adopt the formal proof of the formula $(A \Rightarrow A)$ as presented in Chapter 5 to the case $A = (a \Rightarrow b)$ and get the following.

The formal proof of $((a \Rightarrow b) \Rightarrow (a \Rightarrow b))$ in S is a sequence

$$B_1,\ B_2,\ B_3,\ B_4, B_5$$

such that

$B_1 = (((a \Rightarrow b) \Rightarrow (((a \Rightarrow b) \Rightarrow (a \Rightarrow b)) \Rightarrow (a \Rightarrow b))) \Rightarrow (((a \Rightarrow b) \Rightarrow ((a \Rightarrow b) \Rightarrow (a \Rightarrow b))) \Rightarrow ((a \Rightarrow b) \Rightarrow (a \Rightarrow b))))$,
axiom A2 for $A = (a \Rightarrow b)$, $B = ((a \Rightarrow b) \Rightarrow (a \Rightarrow b))$, and $C = (a \Rightarrow b)$

$B_2 = ((a \Rightarrow b) \Rightarrow (((a \Rightarrow b) \Rightarrow (a \Rightarrow b)) \Rightarrow (a \Rightarrow b)))$,
axiom A1 for $A = (a \Rightarrow b)$, $B = ((a \Rightarrow b) \Rightarrow (a \Rightarrow b))$

$B_3 = (((a \Rightarrow b) \Rightarrow ((a \Rightarrow b) \Rightarrow (a \Rightarrow b))) \Rightarrow ((a \Rightarrow b) \Rightarrow (a \Rightarrow b)))$,
MP application to B_1 and B_2

$B_4 = ((a \Rightarrow b) \Rightarrow ((a \Rightarrow b) \Rightarrow (a \Rightarrow b)))$,
axiom A1 for $A = (a \Rightarrow b)$, $B = (a \Rightarrow b)$

$B_5 = ((a \Rightarrow b) \Rightarrow (a \Rightarrow b))$
MP application to B_3 and B_4

Solution 2

The formula $((a \Rightarrow b) \Rightarrow (a \Rightarrow b))$ is a particular case of a formula $(A \Rightarrow A)$, for $A = (a \Rightarrow b)$

The formal proof of $(A \Rightarrow A)$ in S, as presented in Chapter 5, is a sequence

$$B_1,\ B_2,\ B_3,\ B_4, B_5$$

as defined below.

$B_1 = ((A \Rightarrow ((A \Rightarrow A) \Rightarrow A)) \Rightarrow ((A \Rightarrow (A \Rightarrow A)) \Rightarrow (A \Rightarrow A))),$
axiom A2 for $A = A$, $B = (A \Rightarrow A)$, and $C = A$

$B_2 = (A \Rightarrow ((A \Rightarrow A) \Rightarrow A)),$
axiom A1 for $A = A$, $B = (A \Rightarrow A)$

$B_3 = ((A \Rightarrow (A \Rightarrow A)) \Rightarrow (A \Rightarrow A))),$
MP application to B_1 and B_2

$B_4 = (A \Rightarrow (A \Rightarrow A)),$
axiom A1 for $A = A$, $B = A$

$B_5 = (A \Rightarrow A)$
MP application to B_3 and B_4

PROBLEM 6 (10pts)

Let **GL** be the Gentzen style proof system for classical logic defined in chapter 6.

Prove, by constructing a proper decomposition tree that

$$\vdash_{\mathbf{GL}} ((\neg(a \cap b) \Rightarrow b) \Rightarrow (\neg b \Rightarrow (\neg a \cup \neg b)))$$

Solution

Consider the following tree.

$$\mathbf{T}_{\rightarrow A}$$

$$\longrightarrow ((\neg(a \cap b) \Rightarrow b) \Rightarrow (\neg b \Rightarrow (\neg a \cup \neg b)))$$

$$\mid (\to\Rightarrow)$$

$$(\neg(a \cap b) \Rightarrow b) \longrightarrow (\neg b \Rightarrow (\neg a \cup \neg b))$$

$$\mid (\to\Rightarrow)$$

$$\neg b, (\neg(a \cap b) \Rightarrow b) \longrightarrow (\neg a \cup \neg b)$$

$$\mid (\to \cup)$$

$$\neg b, (\neg(a \cap b) \Rightarrow b) \longrightarrow \neg a, \neg b$$

$$\mid (\to \neg)$$

$$b, \neg b, (\neg(a \cap b) \Rightarrow b) \longrightarrow \neg a$$

$$\mid (\to \neg)$$

$$b, a, \neg b, (\neg(a \cap b) \Rightarrow b) \longrightarrow$$

$$\mid (\neg \to)$$

$$b, a, (\neg(a \cap b) \Rightarrow b) \longrightarrow b$$

$$\bigwedge (\Rightarrow\to$$

$b, a \longrightarrow \neg(a \cap b), b$	$b, a, b \longrightarrow b$
$\mid (\to \neg)$	*axiom*
$b, a, (a \cap b) \longrightarrow b$	
$\mid (\cap \to)$	
$b, a, a, b \longrightarrow b$	
axiom	

All leaves of the decomposition tree are axioms, hence the proof has been found.