CSE371 MIDTERM 1 SOLUTIONS Fall 2018

PART 1: DEFINITIONS

D1 Given a language $\mathcal{L}_{\{\Rightarrow,\cup,\cap,\neg\}}$ and a formula A of this language.

$\models A$ if and only if $v \models A$ for all **truth assignment** $v : VAR \longrightarrow \{T, F\}$

D2 Given formula $A \in \mathcal{F}$ of $\mathcal{L}_{\{\Rightarrow,\cup,\cap,\neg\}}$.

Write definition of v is a restricted model for A.

A **restricted MODEL** for the formula A is any function $w : VAR_A \longrightarrow \{T, F\}$ such that $w^*(A) = T$, where

VAR_A is the sent of all propositional variables appearing in A.

D3 Given a proof system $S = (\mathcal{L}, \mathcal{E}, LA, \mathcal{R})$ and an expression $E \in \mathcal{E}$.

$\vdash_S E$ if and only if there is a sequence $E_1, E_2, , E_n$ of expressions from $\mathcal{E}$, such that $n \geq 1$, and for each $1 < i \leq n$, either $E_i \in LA$ or E_i is a **direct consequence** of some of the preceding expressions in $E_1, E_2, , E_n$ by virtue of one of the rules of inference $r \in \mathcal{R}$.

D4 A proof system $S = (\mathcal{L}, \mathcal{E}, LA, \mathcal{R})$ is **complete** under a semantics **M** if and only if the following holds for any expression $E \in \mathcal{E}$.

$$\vdash_S E \quad \text{if and only if} \quad \models_M E.$$

D5 Write definition: A non-empty set $\mathcal{G} \subseteq \mathcal{F}$ **consistent** under classical semantics.

A non-empty set $\mathcal{G} \subseteq \mathcal{F}$ of **formulas** is called **consistent** if and only if $\mathcal{G}$ **has a model**.

We can also say:

$\mathcal{G} \subseteq \mathcal{F}$ is **consistent** if and only if there is a truth assignment v such that $v \models \mathcal{G}$,

or we say:

$\mathcal{G} \subseteq \mathcal{F}$ is **consistent** if and only if there is v is such that $v^*(A) = T$ for all $A \in \mathcal{G}$.

PART 2: PROBLEMS

PROBLEM 1

Given a set of formulas

$$\mathcal{G} = \{((a \Rightarrow a \cup b)),\ (a \cup b),\ \neg b,\ (c \Rightarrow b)\}$$

1. Show that $\mathcal{G}$ is CONSISTENT under classical semantics. Use shorthand notation.

Solution: We find a restricted model for $\mathcal{G}$. The formula $((a \Rightarrow a \cup b))$, hence any v is its model. $\neg b = T$ only if b=F. We evaluate $(a \cup b) = (a \cup F) = T$ only if a=T. Consequently, $(c \Rightarrow b) = (c \Rightarrow F) = T$ only if c=F. Hence, any v, such that a=T, b= T, and c= F is a model for $\mathcal{G}$.

2. Find a formula A that is iINDEPENDENT of $\mathcal{G}$. Must prove it. Use shorthand notation.

Solution: THIS IS MY SOLUTION. THERE ARE MANY OTHERS!

Let A be any atomic formula $d \in VAR - \{a, b, c\}$. Any v, such that a=T, b= T, and c= F, d= T is a model for $\mathcal{G} \cup \{A\}$. Any v, such that a=T, b= T, and c= F, d= F is a model for $\mathcal{G} \cup \{\neg A\}$.

3. Find an infinite number of formulas that are iINDEPENDENT of $\mathcal{G}$. Justify your answer.

Solution: THIS IS MY SOLUTION. THERE ARE MANY OTHERS!

There is countably infinitely many atomic formulas A=d where $d \in VAR - \{a, b, c\}$.

PROBLEM 2

Given a language $\mathcal{L} = \mathcal{L}_{\{\neg, \Rightarrow, \cup, \cap\}}$. We a define a $\mathbf{L}_4$ **semantics** as follows.

Logical values are $F,\ \bot_1,\ \bot_2,\ T$ and they are ordered: $F < \bot_1 < \bot_2 < T$.

The **connectives** are defined as follow

$\neg\bot_1 = \bot_1,\quad \neg\bot_2 = \bot_2,\quad \neg F = T,\quad \neg T = F.$

For any $x, y \in \{F, \bot_1, \bot_2, T\}$, $\quad x \cap y = min\{x,\ y\}$, $\quad x \cup y = max\{x,\ y\}$, and

$$x \Rightarrow y = \begin{cases} \neg x \cup y & \text{if } x > y \\ T & \text{otherwise} \end{cases}$$

1. Write Truth Tables for **implication** and **negation**.

Solution:

$\Rightarrow$	F	$\bot_1$	$\bot_2$	T
F	T	T	T	T
$\bot_1$	$\bot_1$	T	T	T
$\bot_2$	$\bot_2$	$\bot_2$	T	T
T	F	$\bot_1$	$\bot_2$	T

$\neg$	F	$\bot_1$	$\bot_2$	T
	T	$\bot_1$	$\bot_2$	F

2. Prove $\not\models_{\mathbf{L}_4} ((a \Rightarrow b) \Rightarrow (\neg a \cup b))$. Use **shorthand** notation.

Solution: let v be a truth assignment such that $v(a) = v(b) = \bot_1$.
We evaluate $v^*((a \Rightarrow b) \Rightarrow (\neg a \cup b)) = ((\bot_1 \Rightarrow \bot_1) \Rightarrow (\neg\bot_1 \cup \bot_1)) = (T \Rightarrow (\bot_1 \cup \bot_1)) = (T \Rightarrow \bot_1) = \bot_1$.

This proves that v is a **counter-model** for our formula and that $\not\models_{\mathbf{L}_4} ((a \Rightarrow b) \Rightarrow (\neg a \cup b))$.

Observe that there are other counter-models. For example, v such that $v(a) = v(b) = \bot_2$ is also a counter model, as $v^*((a \Rightarrow b) \Rightarrow (\neg a \cup b)) = ((\bot_2 \Rightarrow \bot_2) \Rightarrow (\neg\bot_2 \cup \bot_2)) = (T \Rightarrow (\bot_2 \cup \bot_2)) = (T \Rightarrow \bot_2) = \bot_2$.

3. Prove that the equivalence defining $\cup$ in terms of negation and implication in classical logic **does not hold** under $\mathbf{L}_4$, i.e. prove that $(A \cup B) \not\equiv_{\mathbf{L}_4} (\neg A \Rightarrow B)$.

Solution: any v such that $v^*(A) = \bot_2$ and $v^*(B) = \bot_1$ is a **counter- model**. This is not the only counter-model.

PROBLEM 3

Consider the Hilbert system $H1 = (\ \mathcal{L}_{\{\Rightarrow\}},\ \ \mathcal{F},\ \ \{A1, A2\},\ \ (MP)\ \frac{A\ ;\ (A \Rightarrow B)}{B}\ )$ where for any $A, B \in \mathcal{F}$

$A1;\ \ (A \Rightarrow (B \Rightarrow A)),\qquad A2:\ \ ((A \Rightarrow (B \Rightarrow C)) \Rightarrow ((A \Rightarrow B) \Rightarrow (A \Rightarrow C)))$.

1. We have proved that the **Deduction Theorem** holds for $H1$.

Use **Deduction Theorem** to prove $(A \Rightarrow (C \Rightarrow B)) \vdash_H (C \Rightarrow (A \Rightarrow B))$.

Solution

We apply the **Deduction Theorem** twice, i.e. we get

$(A \Rightarrow (C \Rightarrow B)) \vdash_H (C \Rightarrow (A \Rightarrow B))$ if and only if

$(A \Rightarrow (C \Rightarrow B)),\ C \vdash_H (A \Rightarrow B)$ if and only if

$(A \Rightarrow (C \Rightarrow B)),\ C,\ A \vdash_H B$

We now construct a proof of $(A \Rightarrow (C \Rightarrow B)), C, A \vdash_H B$ as follows

B_1 : $(A \Rightarrow (C \Rightarrow B))$ hypothesis

B_2 : C hypothesis

B_3 : A hypothesis

B_4 : $(C \Rightarrow B)$ B_1 , B_3 and (MP)

B_5 : C B_2 , B_4 and (MP)

2. Explain why **1.** proves that $(\neg a \Rightarrow ((b \Rightarrow \neg a) \Rightarrow b)) \vdash_H ((b \Rightarrow \neg a) \Rightarrow (\neg a \Rightarrow b))$.

Solution This is **1.** for $A = \neg a,\ C = (b \Rightarrow \neg a)$, and $B = b$.

3. $H1$ **is sound** under classical semantics. Explain why H1 **is not complete**.

Solution The system S **is not complete** under classical semantics means that not all classical tautologies have a proof in S. We have proved that one needs negation and one of other connectives $\cup, \cap, \Rightarrow$ to express all classical connectives, and hence all classical tautologies. Our language contains only implication and one can't express negation in terms of implication alone and hence we can't provide a proof of any tautology i.e. its logically equivalent form in our language $\mathcal{L}_{\{\Rightarrow\}}$.

4. Let $H2$ be the proof system obtained from the system $H1$ by **extending the language** to contain the negation $\neg$ and **adding** one additional axiom:

A3 $((\neg B \Rightarrow \neg A) \Rightarrow ((\neg B \Rightarrow A) \Rightarrow B)))$.

Explain shortly why **Deduction Theorem** holds for H2.

Solution The proof of the Deduction Theorem for H1 used only axioms **A1, A2** so Adding axiom **A3** (and adding $\neg$ to the language) does not change anything in the proof. Hence **Deduction Theorem** holds for H2.

5. We know that $H2$ is **complete**.

Let $H3$ be the proof system obtained from the system $H2$ **adding** additional axiom

A4 $(\neg(A \Rightarrow B) \Rightarrow \neg(A \Rightarrow \neg B))$

Does **Deduction Theorem** holds for $S2$? Justify.

Solution Yes, **it does**, by the same argument as for H2.

Does **Completeness Theorem** holds for $S2$? Justify.

Solution

No, **it does't**. The system S2 is **not sound**. Axiom **A4** is not a tautology.

Any v such that A=T and B=F is a **counter model** for $(\neg(A \Rightarrow B) \Rightarrow \neg(A \Rightarrow \neg B))$.

PROBLEM 4

1 Use the proof system **RS** and its **Completeness Theorem** to prove that

$$\models ((a \cup b) \Rightarrow \neg a) \cup (\neg a \Rightarrow \neg c))$$

Solution

We construct the **decomposition tree** of A as follows

$$\mathbf{T}_A$$

$$((a \cup b) \Rightarrow \neg a)\boxed{\cup}(\neg a \Rightarrow \neg c))$$

$$\mid (\cup)$$

$$((a \cup b)\boxed{\Rightarrow}\neg a), (\neg a \Rightarrow \neg c)$$

$$\mid (\Rightarrow)$$

$$\boxed{\neg}(a\boxed{\cup}b), \neg a, (\neg a \Rightarrow \neg c)$$

$$\bigwedge(\neg\cup)$$

$\neg a, \neg a, (\neg a\boxed{\Rightarrow}\neg c)$	$\neg b, \neg a, (\neg a\boxed{\Rightarrow}\neg c)$
$\mid (\Rightarrow)$	$\mid (\Rightarrow)$
$\neg a, \neg a, \boxed{\neg\neg}a, \neg c$	$\neg b, \neg a, \boxed{\neg\neg}a, \neg c$
$\mid (\neg\neg)$	$\mid (\neg\neg)$
$\neg a, \neg a, a, \neg c$	$\neg b, \neg a, a, \neg c$

All leaves are axioms, so A has a proof in RS.

We know that RS is **complete**, so it proves the

$$\models ((a \cup b) \Rightarrow \neg a) \cup (\neg a \Rightarrow \neg c))$$

2. Use the proof system **RS** to construct a counter model for the formula

$$(((a \Rightarrow b) \cap \neg c) \cup (a \Rightarrow c)).$$

Solution We construct the **decomposition tree** of A as follows

$$\mathbf{T}_A$$

$$(((a \Rightarrow b) \cap \neg c) \cup (a \Rightarrow c))$$

$$\mid (\cup)$$

$$((a \Rightarrow b) \cap \neg c), (a \Rightarrow c)$$

$$\bigwedge (\cap)$$

$$\begin{array}{ccc} (a \Rightarrow b), (a \Rightarrow c) & & \neg c, (a \Rightarrow c) \\ \mid (\Rightarrow) & & \mid (\Rightarrow) \\ \neg a, b, (a \Rightarrow c) & & \neg c, \neg a, c \\ \mid (\Rightarrow) & & \\ \neg a, b, \neg a, c & & \end{array}$$

The **non-axiom leaf** is $L_A : \neg a, \; b, \neg a, \; c.$

Any truth assignment $v : VAR \longrightarrow \{T, F\}$ such that $v(a) = T, \quad v(b) = F, \quad v(c) = F$ **falsifies** the leaf L_A.

By the **strong soundness** of rules of **RS**, v **falsifies** the formula at the root of the decomposition tree $\mathbf{T}_A$, i.e. we proved that

$$\not\models (((a \Rightarrow b) \cap \neg c) \cup (a \Rightarrow c))$$