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LOGIC

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Chapter 7
Introduction to Intuitionistic and Modal Logics

CHAPTER 7 SLIDES

Slides Set 3

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Introduction to Intuitionistic and Modal Logics

Slides Set 3

PART 5: Introduction to Modal Logics
Algebraic Semantics for modal S4 and S5

Introduction to Modal Logics

The **non-classical** logics can be divided in **two groups**: those that **rival classical** logic and those which **extend it**

The **Lukasiewicz**, **Kleene**, and **intuitionistic** logics are in the **first** group

The **modal logics** are in the **second** group

The **rival** logics **do not differ** from classical logic in terms of the **language** employed

The **rival** logics **differ** in that certain **theorems** or **tautologies** of classical logic are rendered **false**, or **not provable** in them

Introduction to Modal Logics

The most **notorious** example of the **rival** difference of logics based on the same **language** is the law of excluded middle

$$(A \cup \neg A)$$

This is **provable** in, and is a **tautology** of **classical** logic

But **is not** provable in, and **is not** tautology of the **intuitionistic** logic

It also **is not** a tautology under any of the **extensional** logics semantics we have discussed

Introduction to Modal Logics

Logics which **extend classical** logic sanction all the theorems of **classical** logic but, generally, **supplement** it in **two** ways

Firstly, the **languages** of these **non-classical** logics are **extensions** of those of **classical** logic

Secondly, the theorems of these **non-classical** logics **supplement** those of **classical** logic

Introduction to Modal Logics

Modal logics are enriched by the addition of two new **connectives** that represent the meaning of expressions "it is necessary that" and "it is possible that"

We use the notation:

I for "it is necessary that" and

C for "it is possible that"

Other notations commonly used are:

∇ , **N**, **L** for "it is necessary that" and

\diamond , **P**, **M** for "it is possible that"

Introduction to Modal Logics

The symbols **N, L, P, M** or alike, are often used in **computer science**

The symbols ∇ and \diamond were **first** to be used in **modal logic** literature

The symbols **I, C** come from **algebraic** and **topological** interpretation of **modal** logics

I corresponds to the topological **interior** of the set and **C** to its **closure**

Introduction to Modal Logics

The idea of a **modal logic** was **first** formulated by an American philosopher, **C.I. Lewis** in **1918**

Lewis has proposed yet another **interpretation** of lasting **consequences**, of the logical **implication**

He created a notion of a **modal truth**, which lead to the notion of **modal logic**

He did it in an **attempt** to avoid, what some felt, the **paradoxes** of semantics for **classical** implication which accepts as **true** that a **false** sentence **implies any sentence**

Introduction to Modal Logics

Lewis' notions appeal to **epistemic** considerations and the whole area of **modal logics** bristles with **philosophical** difficulties and hence the numbers of modal logics have been **created**

Unlike the **classical** connectives, the **modal** connectives **do not** admit of **truth-functional** interpretation, i.e. **do not** accept the **extensional** semantics

This was the **reason** for which **modal** logics were **first** developed as **proof systems**, with intuitive notion of **semantics** expressed by the set of adopted **axioms**

Introduction to Modal Logics

The **first** definition of **modal** semantics, and hence the **proofs** of the **completeness** theorems came some **20 years** later

It took yet another **25 years** for discovery and development of the **second** and more **general** approach to the **modal** semantics

These are the **two established** ways of interpret **modal connectives**, i.e. to define the **modal** semantics

Introduction to Modal Logics

The historically, the **first modal semantics** is due to **McKinsey** and **Tarski (1944, 1946)**

It is a **topological** semantics that provides a powerful **mathematical** interpretation of some of modal logics, namely modal **S4** and **S5**

It connects the **modal** notion of **necessity** with the **topological** notion of the **interior** of a set, and the **modal** notion of **possibility** with the notion of the **closure** of a set

Our **choice** of symbols **I** and **C** for necessity and possibility **connectives** comes from this **interpretation**

The **topological** interpretation mathematically **powerful** as it is, is **less universal** in providing models for **other** modal logics

Introduction to Modal Logics

The most **recent** and the most **general** semantics is due to **Kripke (1964)**. It uses the notion of **possible worlds**.

Roughly, we say that the formula **CA** is **true** if **A** is **true** in **some possible world**, called **actual world**

The formula **IA** is **true** if **A** is **true** in **every possible world**

We **present** here a short version of the **topological** semantics in a form of **algebraic models**

We **leave** the **Kripke semantics** for the reader to **explore** from other, multiple **sources**

Introduction to Modal Logics

As we have already mentioned, **modal** logics were first **developed**, as was the **intuitionistic** logic, in a **form** of **proof systems** only

First Hilbert style **modal** proof system was published by **Lewis** and **Langford** in **1932**

They **presented** a formalization for **two** **modal logics**, which they called **S1** and **S2**

They also **outlined** **three** other proof systems, called **S3**, **S4**, and **S5**

Introduction to Modal Logics

Since then **hundreds** of **modal** logics have been **created**

There are some **standard** books in the subject

These are, **between** the others:

Hughes and **Cresswell (1969)** for **philosophical** motivation for various **modal** logics and **intuitionistic** logic,

Bowen (1979) for a detailed and uniform study of **Kripke models** for **modal** logics,

Segeberg (1971) for excellent modal logics **classification**,
Fitting (1983), for extended and uniform studies of **automated** proof methods for **classes** of **modal** logics

Hilbert Style Modal Proof Systems

Hilbert Style Modal Proof Systems

We present now **Hilbert** style formalization for **S4** and **S5** logics due to **Mc Kinsey** and **Tarski (1948)** and **Rasiowa** and **Sikorski (1964)**

We also **discuss** the **relationship** between **S4** and **S5** , and between the **intuitionistic** logic and **S4** modal logic, as first observed by **Gödel**

The formalizations stress the **connection** between **S4**, **S5** and **topological** spaces which constitute **models** for them

Modal Language

Modal Language

We **add** two extra **one argument** connectives **I** and **C** to the propositional language $\mathcal{L}_{\{U, \cap, \Rightarrow, \neg\}}$, i.e. we adopt

$$\mathcal{L} = \mathcal{L}_{\{U, \cap, \Rightarrow, \neg, \mathbf{I}, \mathbf{C}\}}$$

as the **modal** language. We **read** a formulas **IA**, **CA** as **necessary A** and **possible A**, respectively

The language $\mathcal{L}_{\{U, \cap, \Rightarrow, \neg, \mathbf{I}, \mathbf{C}\}}$ is **common** to all **modal** logics

Modal logics differ on a **choice** of **axioms** and **rules** of inference, when studied as **proof systems** and on a **choice** of respective **semantics**

McKinsey, Tarski Proof Systems

As modal logics **extend** the classical logic, any modal logic contains **two groups** of axioms: **classical** and **modal**

McKinsey, Tarski (1948)

AG1 **classical axioms**

We **adopt** as classical axioms any **complete** set of axioms under classical semantics

AG2 **modal axioms**

M1 **$(IA \Rightarrow A)$**

M2 **$(I(A \Rightarrow B) \Rightarrow (IA \Rightarrow IB))$**

M3 **$(IA \Rightarrow IIA)$**

M4 **$(CA \Rightarrow ICA)$**

Modal S4 and S5

Rules of inference

$$(MP) \frac{A ; (A \Rightarrow B)}{B}, \quad \text{and} \quad (I) \frac{A}{\Box A}$$

The modal rule **(I)** was introduced by Gödel and is referred to as a **necessitation** rule

We define **modal** proof systems **S4** and **S5** as follows

$$S4 = (\mathcal{L}, \mathcal{F}, \text{classical axioms}, M1 - M3, (MP), (I))$$

$$S5 = (\mathcal{L}, \mathcal{F}, \text{classical axioms}, M1 - M4, (MP), (I))$$

Modal S4 and S5

Observe that the **axioms** of **S5** **extend** the axioms of **S4** and both system **share** the same **inference rules**, hence we have immediately the following

Fact For any formula $A \in \mathcal{F}$,

if $\vdash_{S4} A$, then $\vdash_{S5} A$

Rasiowa, Sikorski Proof Systems

It is often the case, as it is for **S4** and **S5**, that **modal connectives** are **definable** by each other

We define them as follows

$$\mathbf{IA} = \neg\mathbf{C}\neg A, \quad \text{and} \quad \mathbf{CA} = \neg\mathbf{I}\neg A$$

Language

We hence assume now that the language \mathcal{L} of **Rasiowa, Sikorski** modal proof systems contains only **one modal connective**

We **choose** it to be **I** and adopt the following language

$$\mathcal{L} = \mathcal{L}_{\{\neg, \cup, \Rightarrow, \mathbf{I}\}}$$

There are, as before, **two groups** of axioms: **classical** and **modal**

Rasiowa, Sikorski Proof Systems

Rasiowa, Sikorski (1964)

AG1 **classical axioms**

We **adopt** as classical axioms any **complete** set of axioms under classical semantics

AG2 **modal axioms**

$$R1 \quad ((IA \cap IB) \Rightarrow I(A \cap B))$$

$$R2 \quad (IA \Rightarrow A)$$

$$R3 \quad (IA \Rightarrow IIA)$$

$$R4 \quad I(A \cup \neg A)$$

$$R5 \quad (\neg I\neg A \Rightarrow I\neg I\neg A)$$

Modal RS4 and RS5

Rules of inference

We adopt the **Modus Ponens** and an additional rule **(RI)**

$$(MP) \frac{A ; (A \Rightarrow B)}{B} \quad \text{and} \quad (RI) \frac{(A \Rightarrow B)}{(IA \Rightarrow IB)}$$

We define modal proof systems **RS4** and **RS5** as follows

$$RS4 = (\mathcal{L}, \mathcal{F}, \text{classical axioms}, R1 - R4, (MP), (RI))$$

$$RS5 = (\mathcal{L}, \mathcal{F}, \text{classical axioms}, R1 - R5, (MP), (RI))$$

Modal RS4 and RS5

Observe that the **axioms** of **RS5** **extend** the axioms of **RS4** and both systems **share** the same inference rules, hence we have immediately the following

Fact For any formula $A \in \mathcal{F}$,

if $\vdash_{RS4} A$, then $\vdash_{RS5} A$

Algebraic Semantics for S4 and S5

Algebraic Semantics for S4 and S5

The McKinsey, Tarski proof systems **S4**, **S5** and Rasiowa, Sikorski proof systems **RS4**, **RS5** are **complete** with the respect to **both topological** semantics, and **Kripke** semantics

We shortly discuss the **topological** semantics, and **algebraic completeness** theorems

We leave the **Kripke semantics** for the reader to **explore** from other, multiple **sources**

Algebraic Semantics for S4 and S5

The **topological semantics** was initiated by McKinsey and Tarski in 1946, 1948 and consequently developed into a field of **Algebraic Logic**

The **algebraic** approach to logic is presented in detail in now **classic** algebraic logic books:

"Mathematics of Metamathematics", Rasiowa, Sikorski (1964),

"An Algebraic Approach to Non-Classical Logics", Rasiowa (1974)

We want to point out that the **first idea** of a connection between **modal** propositional logic and **topology** is due to Tang Tsao -Chen, (1938) and Dugunji (1940)

Algebraic Semantics for S4 and S5

Here are some basic definitions

Boolean Algebra

An abstract algebra $\mathcal{B} = (B, 1, 0, \Rightarrow, \cap, \cup, \neg)$ is said to be a **Boolean algebra** if it is a **distributive lattice** and every element $a \in B$ has a complement $\neg a \in B$

Topological Boolean algebra

By a topological Boolean algebra we mean an abstract algebra

$$\mathcal{B} = (B, 1, 0, \Rightarrow, \cap, \cup, \neg, I)$$

where $(B, 1, 0, \Rightarrow, \cap, \cup, \neg)$ is a **Boolean algebra** and, moreover, the following conditions hold for any $a, b \in B$

$$I(a \cap b) = Ia \cap Ib, \quad Ia \cap a = Ia, \quad IIa = Ia, \quad \text{and} \quad I1 = 1$$

Algebraic Semantics for S4 and S5

The element la is called a **interior** of a

The element $\neg I\neg a$ is called a **closure** of a and will be **denoted** by Ca

Thus the operations I and C are such that

$$Ca = \neg I\neg a \quad \text{and} \quad Ia = \neg C\neg a$$

In this case we write the **topological Boolean algebra** as

$$\mathcal{B} = (B, 1, 0, \Rightarrow, \cap, \cup, \neg, I, C)$$

It is easy to prove that in in any topological Boolean algebra the following **conditions** hold for any $a, b \in B$

$$C(a \cup b) = Ca \cup Cb, \quad Ca \cup a = Ca, \quad CCa = Ca \quad \text{and} \quad C0 = 0$$

Algebraic Semantics for S4 and S5

Example

Let X be a topological space with an **interior** operation I
Then the family $\mathcal{P}(X)$ of all subsets of X is a **topological Boolean algebra** with $1 = X$, with
the operation \Rightarrow defined by the formula

$$Y \Rightarrow Z = (X - Y) \cup Z \text{ for all subsets } Y, Z \text{ of } X$$

and with **set-theoretical operations** of union, intersection, complementation, and the **interior** operation I

Every **sub algebra** of this algebra is a **topological Boolean algebra**, called a **topological field of sets** or, more precisely, a **topological field** of subsets of X

Algebraic Semantics for S4 and S5

Given a topological Boolean algebra

$$(B, 1, 0, \Rightarrow, \cap, \cup, \neg)$$

The element $a \in B$ is said to be **open** (**closed**)
if $a = Ia$ ($a = Ca$)

Clopen Topological Boolean Algebra

A topological Boolean algebra

$$\mathcal{B} = (B, 1, 0, \Rightarrow, \cap, \cup, \neg, I, C)$$

such that every **open** element is **closed** and every **closed** element is **open**, i.e. such that for any $a \in B$

$$Cla = Ia \quad \text{and} \quad ICa = Ca$$

is called a **clopen topological Boolean algebra**

S4, S5 Tautology

We loosely say that a formula A is a modal **S4 tautology** if and only if any **topological Boolean** algebra is a **model** for A

We say that A is a modal **S5 tautology** if and only if any **clopen topological Boolean** algebra is a **model** for A

We put it formally as follows

Modal Algebraic Model

Modal Algebraic Model

For any formula A of a modal language $\mathcal{L}_{\{U, \cap, \Rightarrow, \neg, I, C\}}$ and for any topological Boolean algebra

$$\mathcal{B} = (B, 1, 0, \Rightarrow, \cap, \cup, \neg, I, C)$$

the algebra \mathcal{B} is a **model** for the formula A and denote it by

$$\mathcal{B} \models A$$

if and only if $v^*(A) = 1$ holds for all variables assignments $v : VAR \rightarrow B$

S4, S5 Tautology

Definition of S4 Tautology

A formula A is a modal **S4 tautology** and is denoted by

$$\models_{S4} A$$

if and only if for all **topological Boolean** algebras \mathcal{B} we have that

$$\mathcal{B} \models A$$

Definition of S5 Tautology

A formula A is a modal **S5 tautology** and is denoted by

$$\models_{S5} A$$

if and only if for all **clopen topological Boolean** algebras \mathcal{B} we have that

$$\mathcal{B} \models A$$

S4, S5 Completeness Theorem

We write $\vdash_{S4} A$ and $\vdash_{S5} A$ to denote **provability** in any proof system for modal **S4, S5** logics and in particular the proof systems defined here

Completeness Theorem

For any formula A of the modal language $\mathcal{L}_{\{U, \Box, \Rightarrow, \neg, I, C\}}$

$\vdash_{S4} A$ if and only if $\models_{S4} A$

$\vdash_{S5} A$ if and only if $\models_{S5} A$

The completeness for **S4, S4** follows directly from the following general Algebraic Completeness Theorems

S4 Algebraic Completeness Theorem

S4 Algebraic Completeness Theorem

For any formula A of the modal language $\mathcal{L}_{\{U, \cap, \Rightarrow, \neg, I, C\}}$ the following conditions are equivalent

(i) $\vdash_{S4} A$

(ii) $\models_{S4} A$

(iii) A is valid in every topological field of sets $\mathcal{B}(X)$

(iv) A is valid in every topological Boolean algebra \mathcal{B} with at most 2^{2^r} elements, where r is the number of all subformulas of A

(iv) $v^*(A) = X$ for every variable assignment v in the topological field of sets $\mathcal{B}(X)$ of all subsets of a dense-in-itself metric space $X \neq \emptyset$ (in particular of an n -dimensional Euclidean space X)

S4 Algebraic Completeness Theorem

S5 Algebraic Completeness Theorem

For any formula A of the modal language $\mathcal{L}_{\{\cup, \cap, \Rightarrow, \neg, \mathbf{I}, \mathbf{C}\}}$ the following conditions are equivalent

(i) $\vdash_{S5} A$

(ii) $\models_{S5} A$

(iii) A is valid in every **clopen** topological field of sets $\mathcal{B}(X)$

(iv) A is valid in every **clopen** topological Boolean algebra \mathcal{B} with at most 2^{2^r} elements, where r is the number of all sub formulas of A

S4 and S5 Decidability

The equivalence of conditions **(i)** and **(iv)** of the Algebraic Completeness Theorems proves the **semantical** decidability of modal **S4** and **S5**

S4, S5 Decidability

Any complete **S4, S5** proof system is **semantically decidable**, i.e. the following holds

$$\vdash_{S4} A \text{ if and only if } \mathcal{B} \models A$$

for every topological Boolean algebra \mathcal{B} with at most 2^{2^r} elements, where r is the number of all sub formulas of A

Similarly, we also have

$$\vdash_{S5} A \text{ if and only if } \mathcal{B} \models A$$

for every **clopen** topological Boolean algebra \mathcal{B} with at most 2^{2^r} elements, where r is the number of all sub formulas of A

S4 and S5 Syntactic Decidability

S4, S5 Syntactic Decidability (Wasilewska 1967,1971)

Rasiowa stated in 1950 an **an open problem** of providing a cut-free **RS** type formalization for modal propositional **S4** calculus

Wasilewska solved this open problem in 1967 and presented the result at the **ASL** Summer School and Colloquium in Mathematical Logic, Manchester, August 1969

It appeared in print as *A Formalization of the Modal Propositional S4-Calculus*, **Studia Logica**, North Holland, XXVII (1971)

S4 and S5 Syntactic Decidability

The paper also contained an **algebraic** proof of **completeness** theorem followed by **Gentzen** cut-elimination theorem, the **Hauptsatz**

The resulting **implementation** written in **LISP-ALGOL** was the **first** modal logic **theorem prover** created

It was done with collaboration with **B. Waligorski** and the authors didn't think it to be worth a separate **publication**

Its **existence** was only **mentioned** in the **published** paper

The **S5** Syntactic Decidability follows from the one for **S4** and the following **Embedding Theorems**

Modal S4 and Modal S5

The relationship between **S4** and **S5** was **first** established by **Ohnishi** and **Matsumoto** in **1957-59** and is as follows .

Embedding 1

For any formula $A \in \mathcal{F}$,

$\models_{S4} A$ if and only if $\models_{S5} \mathbf{ICA}$

$\vdash_{S4} A$ if and only if $\vdash_{S5} \mathbf{ICA}$

Embedding 2

For any formula $A \in \mathcal{F}$

$\models_{S5} A$ if and only if $\models_{S4} \mathbf{ICIA}$

$\vdash_{S5} A$ if and only if $\vdash_{S4} \mathbf{ICIA}$

On S4 derivable disjunction

In a **classical** logic it is possible for the disjunction $(A \cup B)$ to be a tautology when **neither** A **nor** B is a tautology

This does not hold for the **intuitionistic** logic. We have a following theorem similar to the **intuitionistic** case to the for modal **S4**

Theorem McKinsey, Tarski (1948)

A disjunction $(IA \cup IB)$ is **S4 provable** if and only if either A or B **S4 provable**, i.e.

$$\vdash_{S4} (IA \cup IB) \quad \text{if and only if} \quad \vdash_{S4} A \quad \text{or} \quad \vdash_{S4} B$$

S4 and Intuitionistic Logic, S5 and Classical Logic

S4 and Intuitionistic Logic

As we have said in the introduction, **Gödel** was the first to consider the **connection** between the **intuitionistic logic** and a logic which was named later **S4**

Gödel's proof was purely **syntactic** in its nature, as the **semantics** for neither **intuitionistic** logic nor modal logic **S4** had not been invented yet

The **algebraic** proof of this fact, was first published by McKinsey and Tarski in **1948**

S4 and Intuitionistic Logic

We define here the **Gödel-Tarski mapping** establishing the **S4** and **intuitionistic** logic connection

We refer the reader to **Rasiowa, Sikorski** book "**Mathematics of Metamathematics**" (1965) for the algebraic proofs of its properties and respective theorems

S4 and Intuitionistic Logic

Let \mathcal{L} be a propositional language of **modal** logic i.e the language

$$\mathcal{L} = \mathcal{L}_{\{\cap, \cup, \Rightarrow, \neg, \Box\}}$$

Let \mathcal{L}_0 be a language obtained from \mathcal{L} by elimination of the connective \Box and by the replacement the **classical** negation connective \neg by the **intuitionistic** negation, which we will **denote** here by a symbol \sim

Such obtained language

$$\mathcal{L}_0 = \mathcal{L}_{\{\cap, \cup, \Rightarrow, \sim\}}$$

is a propositional language of the **intuitionistic** logic

S4 and Intuitionistic Logic

In order to establish the **connection** between the languages

\mathcal{L} and \mathcal{L}_0

and hence between **modal** and **intuitionistic** logic, we consider a **mapping** f which to every formula $A \in \mathcal{F}_0$ of \mathcal{L}_0 **assigns** a formula $f(A) \in \mathcal{F}$ of \mathcal{L}

We define the **mapping** f as follows

Gödel - Tarski Mapping

Definition of Gödel-Tarski mapping

A function

$$f : \mathcal{F}_0 \rightarrow \mathcal{F}$$

such that

$$f(a) = \mathbf{I}a \quad \text{for any } a \in \text{VAR}$$

$$f((A \Rightarrow B)) = \mathbf{I}(f(A) \Rightarrow f(B))$$

$$f((A \cup B)) = (f(A) \cup f(B))$$

$$f((A \cap B)) = (f(A) \cap f(B))$$

$$f(\sim A) = \mathbf{I}\neg f(A)$$

where A, B are any formulas in \mathcal{L}_0 is called a **Gödel-Tarski mapping**

Example

Example

Let A be a formula

$$((\sim A \cap \sim B) \Rightarrow \sim (A \cup B))$$

and f be the Gödel-Tarski mapping. We evaluate $f(A)$ as follows

$$\begin{aligned} f((\sim A \cap \sim B) \Rightarrow \sim (A \cup B)) &= \\ I(f(\sim A \cap \sim B) \Rightarrow f(\sim (A \cup B))) &= \\ I((f(\sim A) \cap f(\sim B)) \Rightarrow f(\sim (A \cup B))) &= \\ I((I\neg fA \cap I\neg fB) \Rightarrow I\neg f(A \cup B)) &= \\ I((I\neg A \cap I\neg B) \Rightarrow I\neg(fA \cup fB)) &= \\ I((I\neg A \cap I\neg B) \Rightarrow I\neg(A \cup B)) & \end{aligned}$$

S4 and Intuitionistic Logic

The following theorem established relationship between intuitionistic and modal S4 logics

Theorem

Let f be the Gödel-Tarski mapping

For any formula A of intuitionistic language \mathcal{L}_0 ,

$$\vdash_I A \quad \text{if and only if} \quad \vdash_{S4} f(A)$$

where $I, S4$ denote any proof systems for intuitionistic and S4 logic, respectively

Classical Logic and Modal S5

In order to establish the connection between the modal **S5** and **classical** logics we consider the following **Gödel-Tarski mapping** between the **modal** language $\mathcal{L}_{\{\Box, \cup, \Rightarrow, \neg, \Box\}}$ and its **classical** sub-language $\mathcal{L}_{\{\neg, \cup, \Rightarrow\}}$

With every **classical** formula **A** we associate a **modal** formula $g(A)$ defined by induction on the length of **A** as follows:

$$g(a) = \Box a, \quad g((A \Rightarrow B)) = \Box(g(A) \Rightarrow g(B)),$$

$$g((A \cup B)) = (g(A) \cup g(B)), \quad g((A \cap B)) = (g(A) \cap g(B)),$$

$$g(\neg A) = \Box \neg g(A)$$

Classical Logic and Modal S5

The following theorem establishes **relationship** between **classical** and **S5** logics

Theorem

Let g be the **Gödel-Tarski mapping** between

$$\mathcal{L}_{\{\neg, \cap, \cup, \Rightarrow\}} \quad \text{and} \quad \mathcal{L}_{\{\cap, \cup, \Rightarrow, \neg, \Box\}}$$

For any formula A of $\mathcal{L}_{\{\neg, \cap, \cup, \Rightarrow\}}$,

$$\vdash_H A \quad \text{if and only if} \quad \vdash_{S5} g(A)$$

where H , $S5$ denote any proof systems for **classical** and **S5** modal logic, respectively