LECTURE 4a
Chapter 4 Review

PART 1: DEFINITIONS
PART 2: Problems
PART 1: Definitions from Chapter 4 you have to know
Definition:  Proof System

Definition 1
By a **proof system** we understand a quadruple

\[ S = (\mathcal{L}, \mathcal{E}, \mathcal{LA}, \mathcal{R}) \]

where

- \( \mathcal{L} = \{ \mathcal{A}, \mathcal{F} \} \) is a **language** of \( S \) with a set \( \mathcal{F} \) of formulas
- \( \mathcal{E} \) is a set of **expressions** of \( S \)
- In particular case \( \mathcal{E} = \mathcal{F} \)
- \( \mathcal{LA} \subseteq \mathcal{E} \) is a **non-empty, finite** set of **logical axioms** of \( S \)
- \( \mathcal{R} \) is a **non-empty, finite** set of **rules of inference** of \( S \)
Definition: Sound Rule of Inference

Definition 2
An inference rule

\[
\begin{array}{c}
\frac{P_1 \;;\; P_2 \;;\; \ldots \;;\; P_m}{C}
\end{array}
\]

is sound under a semantics \( M \) if and only if all \( M \)-models of the set \( \{P_1, P_2, \ldots, P_m\} \) of its premisses are also \( M \)-models of its conclusion \( C \).

In particular, in case of extensional propositional semantics when the condition below holds for any truth assignment \( \nu : VAR \longrightarrow LV \):

If \( \nu \models_M \{P_1, P_2, \ldots, P_m\} \), then \( \nu \models_M C \).
Definition: Direct Consequence

Definition 3
For any rule of inference $r \in \mathcal{R}$ of the form

\[
\frac{P_1 ; P_2 ; \ldots ; P_m}{C}
\]

$C$ is called a **direct consequence** of $P_1, \ldots P_m$ by virtue of the rule $r \in \mathcal{R}$
Definition:  Formal Proof

Definition 4

A formal proof of an expression \( E \in \mathcal{E} \) in a proof system \( S = (\mathcal{L}, \mathcal{E}, LA, R) \) is a sequence

\[
A_1, A_2, \ldots, A_n \text{ for } n \geq 1
\]

of expressions from \( \mathcal{E} \), such that

\[
A_1 \in LA, \quad A_n = E
\]

and for each \( 1 < i \leq n \), either \( A_i \in LA \) or \( A_i \) is a direct consequence of some of the preceding expressions by virtue of one of the rules of inference

\( n \geq 1 \) is the length of the proof \( A_1, A_2, \ldots, A_n \)
NOTATION: Provable Expressions

Notation
We write $\vdash_S E$ to denote that $E \in \mathcal{E}$ has a formal proof in the proof system $S$
A set

$$P_S = \{E \in \mathcal{E} : \vdash_S E\}$$

is called the set of **all provable expressions** in $S$
Definition: Sound $S$

Definition 5
Given a proof system

$$S = (L, E, LA, R)$$

We say that the system $S$ is sound under a semantics $M$ iff the following conditions hold

1. Logical axioms $L_A$ are tautologies of under the semantics $M$, i.e.
   $$L_A \subseteq T_M$$

2. Each rule of inference $r \in R$ is sound under the semantics $M$
THEOREMS: Soundness Theorem

Let $P_S$ be the set of all provable expressions of $S$ i.e.

$$P_S = \{ A \in \mathcal{E} : \vdash_S A \}$$

Let $T_M$ be a set of all expressions of $S$ that are tautologies under a semantics $M$, i.e.

$$T_M = \{ A \in \mathcal{E} : \models M A \}$$

Our GOAL is to prove the following theorems:

Soundness Theorem (for $S$ and semantics $M$)

$$P_S \subseteq T_M$$

i.e. for any $A \in \mathcal{E}$, the following implication holds

If $\vdash_S A$ then $\models M A$
THEOREMS: Completeness Theorem

Completeness Theorem (for $S$ and semantics $M$)

$$P_S = T_M$$

i.e. for any $A \in \mathcal{E}$, the following holds

$$\vdash_S A \text{ if and only if } \models_M A$$

The Completeness Theorem consists of two parts:

Part 1: Soundness Theorem

$$P_S \subseteq T_M$$

Part 2: Completeness Part of the Completeness Theorem

$$T_M \subseteq P_S$$
PART 2: Simple Problems
Problem 1
Given a proof system:

\[ S = (\mathcal{L}_{\{\neg,\Rightarrow\}}, \mathcal{F}, \{(A \Rightarrow A), (A \Rightarrow (\neg A \Rightarrow B))\}), \mathcal{R} = \{(r)\} \]

where \((r)\)

\[ \frac{(A \Rightarrow B)}{(B \Rightarrow (A \Rightarrow B))} \]

Write a formal proof in \(S\) with 2 applications of the rule \((r)\)

**Solution:** There are many solutions. Here is one of them.

Required formal proof is a sequence \(A_1, A_2, A_3\), where

\(A_1 = (A \Rightarrow A)\)
(Axiom)

\(A_2 = (A \Rightarrow (A \Rightarrow A))\)
Rule \((r)\) application 1 for \(A = A, B = A\)

\(A_3 = ((A \Rightarrow A) \Rightarrow (A \Rightarrow (A \Rightarrow A)))\)
Rule \((r)\) application 2 for \(A = A, B = (A \Rightarrow A)\)
Given a proof system:

\[ S = (L_{\neg, \Rightarrow}, \mathcal{F}, \{(A \Rightarrow A), (A \Rightarrow (\neg A \Rightarrow B))\}, (r) \frac{(A \Rightarrow B)}{(B \Rightarrow (A \Rightarrow B))}) \]

**Problem 2**
Prove that \( S \) is **sound** under classical semantics.

**Solution**
1. Both axioms of \( S \) are basic classical tautologies
2. Consider the rule of inference of \( S \)

\[ (r) \frac{(A \Rightarrow B)}{(B \Rightarrow (A \Rightarrow B))} \]

Assume that its premise (the only premise) is true, i.e. let \( v \) be any truth assignment, such that \( v^*(A \Rightarrow B) = T \)

We evaluate logical value of the conclusion under the truth assignment \( v \) as follows

\[ v^*(B \Rightarrow (A \Rightarrow B)) = v^*(B) \Rightarrow T = T \]

for any \( B \) and any value of \( v^*(B) \)
Given a proof system:

\[ S = (\mathcal{L}_{\{\neg, \Rightarrow\}}, \mathcal{F}, \{(A \Rightarrow A), (A \Rightarrow (\neg A \Rightarrow B))\}, (r) \frac{(A \Rightarrow B)}{(B \Rightarrow (A \Rightarrow B))}) \]

Problem 3.
Write a \textbf{formal proof} of your choice in \( S \) with 2 applications of the rule \((r)\)

Solution
There many of such proofs, of different length, with different choice if axioms - here is my choice: \( A_1, A_2, A_3 \), where

\( A_1 = (A \Rightarrow A) \)
(Axiom)

\( A_2 = (A \Rightarrow (A \Rightarrow A)) \)

Rule \((r)\) application 1 for \( A = A, B = A \)

\( A_3 = ((A \Rightarrow A) \Rightarrow (A \Rightarrow (A \Rightarrow A))) \)

Rule \((r)\) application 2 for \( A = A, B = (A \Rightarrow A) \)
Formal Proof

Given a proof system:

\[ S = (\mathcal{L}_{\neg, \Rightarrow}, \mathcal{F}, \{(A \Rightarrow A), (A \Rightarrow (\neg A \Rightarrow B))\}, (r) \frac{(A \Rightarrow B)}{(B \Rightarrow (A \Rightarrow B))}) \]

Problem 4

1. Prove, by constructing a formal proof that

\[ \vdash_S ((\neg A \Rightarrow B) \Rightarrow (A \Rightarrow (\neg A \Rightarrow B))) \]

Solution  Required formal proof is a sequence \( A_1, A_2 \), where

\[ A_1 = (A \Rightarrow (\neg A \Rightarrow B)) \]

Axiom

\[ A_2 = (((\neg A \Rightarrow B) \Rightarrow (A \Rightarrow (\neg A \Rightarrow B))) \]

Rule \((r)\) application for \( A = A, B = (\neg A \Rightarrow B) \)
Soundness Theorem

2. Does above point 1. prove that

\[ \vdash ((\neg A \implies B) \implies (A \implies (\neg A \implies B)))? \]

Solution

Yes, it does because the system S is sound and we proved by Mathematical Induction over the length of a proof that if S is sound, then the Soundness Theorem holds for S.
Problem 5
Given a proof system:

\[ S = (\mathcal{L}_{\neg, \Rightarrow}, \mathcal{F}, \{(A \Rightarrow A), (A \Rightarrow (\neg A \Rightarrow B))\}, (r) \frac{(A \Rightarrow B)}{(B \Rightarrow (A \Rightarrow B))}) \]

Prove that \( S \) is \textbf{not sound} under \( K \) semantics

Solution
Axiom \( (A \Rightarrow A) \) is not a \( K \) semantics tautology
Any truth assignment \( v \) such that \( v^*(A) = \bot \) is a \textbf{counter-model} for it
This proves that \( S \) is \textbf{not sound} under \( K \) semantics