cse371/mat371
LOGIC

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LECTURE 3b

Chapter 3
Propositional Semantics: Classical and Many Valued

Many Valued Semantics:
Łukasiewicz, Heyting, Kleene, Bohvar

First Many Valued Logics

The study of many valued logics in general and 3-valued logics in particular has its beginning in the work of a Polish mathematician **Jan Leopold Łukasiewicz** in 1920

Łukasiewicz was the first to **define** a 3 - valued **semantics** for the language

$$\mathcal{L}_{\{\neg,\cap,\cup,\Rightarrow\}}$$

of classical logic, and called it a **logic** for short

He left the problem of **finding** a proper axiomatic proof system for it **open**

First Many Valued Logics

The other 3 - valued **semantics** presented here were also first called logics and this terminology is still widely used

Nevertheless, as these logics were **defined only** semantically, i.e. defined only by providing a semantics for their languages we call them **semantics** (for logics to be developed), not logics

Creating a Logic

Existence of a proper axiomatic proof system for a given **semantics** and **proving** its completeness is always a next open question to be **answered** (when it is possible)

A process of creating a **logic** (based on a given language) is **three fold:** we have to

define semantics,

create axiomatic proof system and

prove completeness theorem that establishes a **relationship** between semantics and proof system

First Many Valued Logics

We present here some of the first 3-valued extensional **semantics**, historically called 3-valued logics

They are **named** after their authors: Łukasiewicz, Kleene, Heyting, and Bochvar

We assume that the **language** of all **semantics** (logics) considered here except of Bochvar semantics is

$$\mathcal{L}_{\{\neg,\ \cup,\ \cap,\ \Rightarrow\}}$$

3-Valued Semantics

All three valued **semantics** considered here enlist a **third** logical value which we **denote** by $\perp$, or m in case of Bochvar semantics

The **third** logical value **denotes** a notion of unknown, uncertain, undefined, or even the notion of we don't have a complete information about depending on the context and **motivation** for the **semantics** (logic)

The symbol $\perp$ is the most frequently used for different concepts of **unknown**

Many Valued Semantics

The **third** value $\perp$ corresponds also to some notion of incomplete information, inconsistent information, or to a notion of being undefined , or unknown

Historically all these **semantics**, and many others were and still are called **logics**

We will also use the name **logic** for them, instead saying each time " logic defined semantically", or "semantics for a given logic"

3 Valued Semantics Assumptions

We **assume** that the third logical value is **intermediate** between truth and falsity, i.e.

the set of logical values is **ordered** and we have the following

Assumption 1

$$F < \perp < T, \quad \text{and} \quad F < m < T$$

Assumption 2

We take T as designated value, i.e. T is the value that **defines** the notions of satisfiability and tautology

Many Valued Extensional Semantics

Formal definition of all many valued semantics presented here follows the **definition** of the extensional semantics **M** in general, and the pattern presented in detail for the **classical semantics** in particular

It consists of giving **definitions** of the following main components:

Step 1: given the language $\mathcal{L}$ we **define** a set of logical values and its distinguish value T and **define** all extensional logical connectives of $\mathcal{L}$

Step 2: we **define** notions of a truth assignment and its extension

Step 3: we **define** notions of satisfaction, model, counter model

Step 4: we **define** notions tautology under the semantics **M**

Łukasiewicz Semantics **L**

Motivation

Łukasiewicz developed his semantics (called logic) to deal with future **contingent** statements

Contingent statements are not just neither **true** nor **false** but are indeterminate in some metaphysical sense

It is not only that we do not know their truth value but rather that they do not possess one

L Semantics: Language

We define **all the steps** in case of Łukasiewicz **semantics** (logic) to establish a pattern and proper notation and leave adopting all steps to the case of other semantics as an **exercise**

Step 1 The **language** is $\mathcal{L}_{\{\neg, \cap, \cup, \Rightarrow\}}$

Observe that the language is the same as in the **classical** semantics case

The set $\mathcal{F}$ of **formulas** is defined in a standard way

L Semantics: Connectives

Step 1 Connectives

We assumed: $F < \perp < T$ and we define the connectoves as follows

Negation $\neg$ is a **function**

$$\neg : \ \{T, \perp, F\} \longrightarrow \{T, \perp, F\}$$

such that $\neg \perp = \perp, \ \ \neg T = F, \ \ \neg F = T$

Conjunction $\cap$ is a **function**

$$\cap : \ \{T, \perp, F\} \times \{T, \perp, F\} \longrightarrow \{T, \perp, F\}$$

such that for any $(x, y) \in \{T, \perp, F\} \times \{T, \perp, F\}$, we put

$$x \cap y = min\{x, \ y\}$$

L Semantics: Connectives

Disjunction $\cup$ is a **function**

$$\cup : \{T, \perp, F\} \times \{T, \perp, F\} \longrightarrow \{T, \perp, F\}$$

such that for any $(a, b) \in \{T, \perp, F\} \times \{T, \perp, F\}$, we put

$$x \cup y = max\{x, \ y\}$$

Implication $\Rightarrow$ is a **function**

$$\Rightarrow : \{T, \perp, F\} \times \{T, \perp, F\} \longrightarrow \{T, \perp, F\}$$

such that for any $(x, y) \in \{T, \perp, F\} \times \{T, \perp, F\}$, we put

$$x \Rightarrow y = \begin{cases} \neg x \cup y & \text{if } x > y \\ T & \text{otherwise} \end{cases}$$

L Connectives Truth Tables

Negation

$\neg$	F	$\perp$	T
	T	$\perp$	F

Conjunction

$\cap$	F	$\perp$	T
F	F	F	F
$\perp$	F	$\perp$	$\perp$
T	F	$\perp$	T

L Connectives Truth Tables

Disjunction

$\cup$	F	$\perp$	T
F	F	$\perp$	T
$\perp$	$\perp$	$\perp$	T
T	T	T	T

Implication

$\Rightarrow$	F	$\perp$	T
F	T	T	T
$\perp$	$\perp$	T	T
T	F	$\perp$	T

L Semantics: Truth Assignment

Step 2 Truth assignment and its extension

Definition

A **truth assignment** is any function

$$v: \ VAR \longrightarrow \{F, \ \perp, \ T\}$$

Observe that the domain of truth assignment is the set of propositional variables, i.e. the truth assignment is defined only for **atomic formulas**

Truth Assignment Extension v^*

Definition

Given a truth assignment $v : VAR \longrightarrow \{T, \perp, F\}$

We define its **extension** $v^* : \mathcal{F} \longrightarrow \{T, \perp, F\}$ by the **induction** on the degree of formulas as follows

(i) for any $a \in VAR$, $v^*(a) = v(a)$;

(ii) and for any $A, B \in \mathcal{F}$ we put

$$v^*(\neg A) = \neg v^*(A);$$

$$v^*((A \cap B)) = v^*(A) \cap v^*(B);$$

$$v^*((A \cup B)) = v^*(A) \cup v^*(B);$$

$$v^*((A \Rightarrow B)) = v^*(A) \Rightarrow v^*(B)$$

L Semantics: Satisfaction Relation

Step 3 Satisfaction, Model, Counter Model

Definition

Let $v : VAR \longrightarrow \{T, \perp F\}$

We say that a truth assignment v **L satisfies** a formula $A \in \mathcal{F}$ if and only if $v^*(A) = T$

Notation: $v \models_L A$

Definition

We say that a truth assignment v **does not L satisfy** a formula $A \in \mathcal{F}$ if and only if $v^*(A) \neq T$

Notation: $v \not\models_L A$

$\mathbf{L}$ Semantics: Model, Counter Model

Model

Any truth assignment $v : VAR \longrightarrow \{F, \perp, T\}$ such that

$$v \models_L A$$

is called a **L model** for A

Counter Model

Any v such that

$$v \not\models_L A$$

is called a **L counter model** for the formula A

L Semantics: Tautology

Step 4 Tautology

For any $A \in \mathcal{F}$,

A is a **L tautology** if and only if $v^*(A) = T$ for all $v : VAR \longrightarrow \{F, \perp, T\}$

We also say that

A is a **L tautology** if and only if all truth assignments $v : VAR \longrightarrow \{F, \perp, T\}$ are **L models** for A

Notation

$$\models_L A$$

L Tautologies

We denote the set of all **L tautologies** by

$$\mathbf{LT} = \{A \in \mathcal{F} : \models_L A\}$$

Let **LT**, **T** be the sets of all **L tautologies** and the **classical** tautologies, respectively.

Q1 Is the L logic (defined semantically!) really **different** from the **classical logic**?

It means are theirs sets of tautologies different?

Answer: YES, they are different sets

We know that

$$\models (\neg a \cup a)$$

We will show that

$$\not\models_L (\neg a \cup a)$$

Classical and **L** Tautologies

Consider the formula $(\neg a \cup a)$

Take a truth assignment v such that

$$v(a) = \perp$$

Evaluate

$v^*(\neg a \cup a) = v^*(\neg a) \cup v^*(a) = \neg v(a) \cup v(a)$

$= \neg \perp \cup \perp = \perp \cup \perp = \perp$

This proves that v is a **counter-model** for $(\neg a \cup a)$, i.e.

$$\not\models_L \ (\neg a \cup a)$$

and we proved

$$\mathbf{LT} \neq \mathbf{T}$$

Classical and **L** Tautologies

Q2 Do the **L** and classical **logics** have something more in common besides the same language?

YES, they also share some tautologies

Q3 Is there relationship (if any) between their sets of **tautologies** **LT** and **T**?

YES, their sets of **tautologies** **LT** and **T** do have an interesting relationship

Classical and **L** Tautologies

Let's restrict the functions defining **L** connectives (Truth Tables for **L** connectives) to the values T and F

Observe that by doing so we get the Truth Tables for classical connectives, i.e. the following holds for any $A \in \mathcal{F}$

If $v^*(A) = T$ for all $v : VAR \longrightarrow \{F, \perp, T\}$,
then $v^*(A) = T$ for all $v : VAR \longrightarrow \{F, T\}$

We have hence **proved** that

$$\mathbf{LT} \subset \mathbf{T}$$

Exercise

Exercise

Use the fact that $v : VAR \longrightarrow \{F, \perp, T\}$ is such that

$$v^*((a \cap b) \Rightarrow \neg b) = \perp$$

under **L** semantics **to evaluate**

$$v^*(((b \Rightarrow \neg a) \Rightarrow (a \Rightarrow \neg b)) \cup (a \Rightarrow b))$$

Use **shorthand** notation.

Exercise

Solution

Observe that $((a \cap b) \Rightarrow \neg b) = \perp$ in two cases

c1: $(a \cap b) = \perp$ and $\neg b = F$

c12: $(a \cap b) = T$ and $\neg b = \perp$

Consider **c1**

We have $\neg b = F$, i.e. $b = T$

Hence $(a \cap T) = \perp$ if and only if $a = \perp$

We get that v is such that $v(a) = \perp$ and $v(b) = T$

Exercise

We got from analyzing case **c1** that v is such that $v(a) = \perp$ and $v(b) = T$

We evaluate $v^*(((b \Rightarrow \neg a) \Rightarrow (a \Rightarrow \neg b)) \cup (a \Rightarrow b)) =$
$(((T \Rightarrow \neg \perp) \Rightarrow (\perp \Rightarrow \neg T)) \cup (\perp \Rightarrow T)) = ((\perp \Rightarrow \perp) \cup T) = T$

Consider **c2**

We have $\neg b = \perp$, i.e. $b = \perp$ and $(a \cap \perp) = T$, what is **impossible**

Hence v from case **c1** is the only one and

$$v^*(((b \Rightarrow \neg a) \Rightarrow (a \Rightarrow \neg b)) \cup (a \Rightarrow b)) = T$$

Łukasiewicz Life, Works and Logics

Jan Leopold Łukasiewicz was born on 21 December 1878 in Lwow, historically a Polish city, at that time the capital of Austrian Galicia

He died on 13 February 1956 in Ireland and is buried in Glasnevin Cemetery in Dublin, " far from dear Lwow and Poland ", as his gravestone reads

Here is a very good, interesting and extended entry in Stanford Encyclopedia of Philosophy about his life, influences, achievements, and logics

http://plato.stanford.edu/entries/lukasiewicz/index.html

Heyting Semantics **H**

Motivation and History

We discuss here the Heyting semantics **H** because of its connection with intuitionistic logic

The **H** connectives are defined as operations on the set $\{F, \perp, T\}$ in such a way that they form a 3-element pseudo-Boolean algebra

Pseudo-Boolean algebras were created by McKinsey and Tarski in 1948 to provide **semantics** for the intuitionistic logic

Pseudo-Boolean algebras are often called Heyting algebras

Motivation and History

The intuitionistic logic, was defined by its inventor Brouwer and his school in 1900s as a proof system only

Heyting provided provided its **first axiomatization** which everybody accepted

McKinsey and Tarski proved in 1942 the **completeness** of the Heyting axiomatization with respect to their **pseudo Boolean** algebras semantics

The **pseudo boolean** algebras are **also** called Heyting algebras in his honor and so is our semantics **H**

Motivation and History

A formula *A* is an intuitionistic tautology if and only if it is true in all pseudo boolean algebras

We prove that the operations defined by **H** connectives form a 3-element **pseudo boolean** algebra

Hence, if *A* is an intuitionistic tautology, it is also a tautology under the 3- valued Heyting semantics

If A **is not** a 3- valued Heyting tautology, then it **is not** an intuitionistic tautology

It means that the 3-valued Heyting semantics is a good candidate for a **counter model** for the formulas that **might not** be intuitionistic tautologies

H Logic and Intuitionistic Logic

Denote by **IT**, **HT** the sets of all **tautologies** of the intuitionistic logic and Heyting 3-valued logic (semantics), respectively .

We have that

$$\mathbf{IT} \subset \mathbf{HT}$$

We conclude that for any formula A,

$$\text{If} \quad \not\models_{\mathbf{H}} A \quad \text{then} \quad \not\models_{\mathbf{I}} A$$

It means that if we show that a formula A has an **H counter model**, then we have proved that A it **is not** an intuitionistic tautology

Kripke Models

The other type of **semantics** for the intuitionistic logic were defined by Kripke in 1964

They are called **Kripke models**

The Kripke models were later proved to be **equivalent** to the pseudo boolean algebras models in case of the intuitionistic logic

Kripke models also provide a general method of defining semantics for many classes of logics

That includes **semantics** for various modal logics and new logics developed and being developed by computer scientists

H Semantics

Language

$$\mathcal{L}_{\{\neg, \Rightarrow, \cup, \cap\}}$$

Connectives

$\cup$ and $\cap$ are the same as in the case of **Ł** semantics, i.e. for any $(x, y) \in \{T, \perp, F\} \times \{T, \perp, F\}$ we put

$$x \cup y = max\{x,\ y\}, \qquad x \cap y = min\{x,\ y\}$$

where $F < \perp < T$

H Semantics

Implication

$$\Rightarrow: \quad \{T, \perp, F\} \times \{T, \perp, F\} \longrightarrow \{T, \perp, F\}$$

such that for any $(x, y) \in \{T, \perp, F\} \times \{T, \perp, F\}$ we put

$$x \Rightarrow y = \begin{cases} T & \text{if } x \leq y \\ y & \text{otherwise} \end{cases}$$

Negation

$$\neg x = x \Rightarrow F$$

H Truth Tables

Implication

$\Rightarrow$	F	$\perp$	T
F	T	T	T
$\perp$	F	T	T
T	F	$\perp$	T

Negation

$\neg$	F	$\perp$	T
	T	*F*	F

Sets of Tautologies Relationships

HT, **T**, **LT** denote the set of all tautologies of the **H**, classical, and **L** semantics, respectively

Relationships

$$\mathbf{HT} \neq \mathbf{T} \neq \mathbf{LT}$$

$$\mathbf{HT} \subset \mathbf{T}$$

Proof of $\mathbf{HT} \neq \mathbf{T}$

For the formula $(\neg a \cup a)$ we have:

$$\models (\neg a \cup a) \quad \text{and} \quad \not\models_{\mathbf{H}} (\neg a \cup a)$$

Sets of Tautologies Relationships

Proof of **HT $\neq$ LT**

Take a truth assignment v such that

$$v(a) = v(b) = \perp$$

We verify that

$$\not\models_{\mathbf{H}} (\neg(a \cap b) \Rightarrow (\neg a \cup \neg b))$$

and

$$\models_{\mathbf{L}} (\neg(a \cap b) \Rightarrow (\neg a \cup \neg b))$$

Sets of Tautologies Relationships

Proof of **HT** $\subset$ **T**

Observe that if we restrict the truth tables for **H** connectives to logical values T and F only we get the truth tables for the classical connectives , i.e. and the following holds for any formula A

If $v^*(A) = T$ for all $v: VAR \longrightarrow \{F, \perp, T\}$,

then $v^*(A) = T$ for all $v: VAR \longrightarrow \{F, T\}$

All together we have **proved** that the classical semantics **extends** both **L** and **H** semantics, i.e.

LT $\subset$ **T** and **HT** $\subset$ **T**

Kleene Semantics **K**

Motivation

Kleene's semantics was originally conceived to accommodate **undecided** mathematical statements

It models a situation where the third logical value $\perp$ intuitively represents the notion of "undecided", or "state of partial ignorance"

A sentence is **assigned** a value $\perp$ just in case it is **not known** to be either true or false

Kleene Semantics **K**

For **example** imagine a detective trying to solve a **murder**

He may **conjecture** that Jones killed the **victim**

He cannot, at present, **assign** a truth value T or F to his conjecture, so we **assign** the value $\perp$

But it is certainly either true or false and hence $\perp$ represents our **ignorance** rather then total **unknown**

Kleene Semantics **K**

Language

We adopt the same language as in a case of classical, Łukasiewicz's **L**, and Heyting **H** semantics, i.e.

$$\mathcal{L}_{\{\neg,\Rightarrow,\cup,\cap\}}$$

Connectives

We assume, as before, that $F < \perp < T$

The connectives $\neg, \cup, \cap$ of **K** are defined as in **L, H** semantics, i.e.

$$\neg \perp = \perp, \quad \neg F = T, \quad \neg T = F$$

and for any $(x, y) \in \{T, \perp, F\} \times \{T, \perp, F\}$ we put

$$x \cup y = max\{x, \ y\}$$

$$x \cap y = min\{x, \ y\}$$

K Semantics: Connectives

K Implication

Kleene's implication **differ** from **L** and **H** semantics

The **K** implication is defined by the same formula as the classical, i.e. for any $(x, y) \in \{T, \bot, F\} \times \{T, \bot, F\}$

$$x \Rightarrow y = \neg x \cup b$$

The connectives **truth tables** for the **K** negation, disjunction and conjunction are the same as the tables for **L, H**

K implication table is

$\Rightarrow$	F	$\bot$	T
F	T	T	T
$\bot$	$\bot$	$\bot$	T
T	F	$\bot$	T

K Semantics: Tautologies

Set of all **K** tautologies is

$$\mathbf{KT} = \{A \in \mathcal{F} : \models_{\mathbf{K}} A\}$$

Relationship between **Ł, H, K**, and classical semantics is

$$\mathbf{LT} \neq \mathbf{KT}, \quad \mathbf{HT} \neq \mathbf{KT}, \quad \text{and} \quad \mathbf{KT} \subset \mathbf{T}$$

Proof Obviously $\models_L (a \Rightarrow a)$ and $\models (a \Rightarrow a)$ We take v such that $v(a) = \perp$ and evaluate in **K** semantics

$v^*(a \Rightarrow a) = (v(a) \Rightarrow v(a)) = (\perp \Rightarrow \perp) = \perp$

This **proves** that $\not\models_{\mathbf{K}} (a \Rightarrow a)$ and hence

$$\mathbf{LT} \neq \mathbf{KT} \quad \text{and} \quad \mathbf{LT} \neq \mathbf{KT}$$

K Tautologies

The third property

$$\mathbf{KT} \subset \mathbf{T}$$

follows directly from the the fact that, as in the **L , H** case, if we **restrict** the **K** connectives definitions functions to the values T and F only we get the functions defining the classical connectives

All together we have **proved** that the classical semantics **extends** all three **L , H** and **K** semantics, i.e.

$$\mathbf{LT} \subset \mathbf{T}, \quad \mathbf{HT} \subset \mathbf{T}, \quad \text{and} \quad \mathbf{K} \subset \mathbf{T}$$

L, H, K Decidability

Verification and Decidability

The following theorem justifies the correctness of the **truth table** method of **tautology verification** for for **L, H, K** semantics

Theorem 1

For any formula A of $\mathcal{L}_{\{\neg,\Rightarrow,\cup,\cap\}}$, for any $\mathbf{M} \in \{\mathbf{L}, \mathbf{H}, \mathbf{K}\}$

$$\models_{\mathbf{M}} A \quad \text{if and only if} \quad v_A \models_{\mathbf{M}} A$$

$$\text{for all} \quad v_A : VAR_A \longrightarrow \{T, \perp, F\}$$

We also say that

$\models_{\mathbf{M}} A$ if and only if all v_A are **restricted** $\mathbf{M}$ models for A, and $\mathbf{M} \in \{\mathbf{L}, \mathbf{H}, \mathbf{K}\}$

L, H, K Decidability

The following theorem proves the decidability of the tautology **verification** procedure for **L, H, K** semantics

Theorem 2

For any formula A of $\mathcal{L}_{\{\neg,\Rightarrow,\cup,\cap\}}$, one has to **examine** at most 3^{VAR_A} truth assignments $v_A : VAR_A \longrightarrow \{F, \perp, T\}$ in order to **decide** whether

$$\models_{\mathbf{M}} A \quad \text{or} \quad \not\models_{\mathbf{M}} A$$

i.e. the notion of **M** tautology is **decidable**

for any semantics $\mathbf{M} \in \{\mathbf{L}, \mathbf{H}, \mathbf{K}\}$

Proofs of **Theorems 1, 2** are carried in the same way as in case of classical semantics and are left as an exercise

K Tautologies Revisited

Exercise

We know that formulas

$$((a \cap b) \Rightarrow a), \quad (a \Rightarrow (a \cup b)), \quad (a \Rightarrow (b \Rightarrow a))$$

are classical tautologies

Show that **none** of them is **K** tautology

Solution

Consider any v such that $v(a) = v(b) = \perp$

We evaluate (in short hand notation)

$v^*(((a \cap b) \Rightarrow a) = (\perp \cap \perp) \Rightarrow \perp = \perp \Rightarrow \perp = \perp$

K Tautologies Revisited

$v^*((a \Rightarrow (a \cup b))) = \perp \Rightarrow (\perp \cup \perp) = \perp \Rightarrow \perp = \perp$ and

$v^*((a \Rightarrow (b \Rightarrow a))) = (\perp \Rightarrow (\perp \Rightarrow \perp) = \perp \Rightarrow \perp = \perp$

This proves that any v such that

$$v(a) = v(b) = \perp$$

is a **counter model** for all of them

We **generalize** this example and **prove** (by induction over the degree of a formula) that a truth assignment v such that

$$v(a) = \perp \quad \text{for all} \quad a \in VAR$$

is a **counter model** for **any formula** A of $\mathcal{L}_{\{\neg, \Rightarrow, \cup, \cap\}}$

$\mathbf{K}$ Tautologies Revisited

We proved the following

Theorem

For any formula A of $\mathcal{L}_{\{\neg,\Rightarrow,\cup,\cap\}}$, $\not\models_{\mathbf{K}} A$

In particular, the set of all **K tautologies** is empty, i.e.

$$\mathbf{KT} = \emptyset$$

Observe that the **Theorem** does not invalidate relationships

$$\mathbf{LT} \neq \mathbf{KT}, \quad \mathbf{HT} \neq \mathbf{KT}, \quad \text{and} \quad \mathbf{KT} \subset \mathbf{T}$$

between **Ł, H, K**, and classical semantics

They become now perfectly true statements

$$\mathbf{LT} \neq \emptyset, \quad \mathbf{T} \neq \emptyset, \quad \text{and} \quad \emptyset \subset \mathbf{T}$$

K Tautologies Revisited

When we develop a new logic by defining its **semantics** we must make sure for the semantics to be such that it has a non empty set of its **tautologies**

This is why we adopted (**Set 2**) the following definition

Definition

Given a language $\mathcal{L}_{CON}$ and its semantics **M**

We say that the semantics **M** is **well defined** if and only if its set **MT** of all tautologies is non empty, i.e. when

$$\mathbf{MT} \neq \emptyset$$

K Tautologies Revisited

The semantics **K** is an example of a **correctly** and **carefully** defined semantics that **is not** well defined in terms of the above definition

Obviously the semantics **L** and **H** are well defined

We write is as a following separate fact

K Tautologies Revisited

Fact

The semantics **L** and **H** are **well defined**, but the Kleene semantics **K is not**

K semantics also provides a justification for a need of introducing a **distinction** between correctly and well defined semantics

This is the main reason, beside its historical value, why it is included here

Bochvar Semantics **B**

Motivation

Consider a **semantic paradox** given by a sentence:

this sentence is false.

If it is true it must be false,

if it is false it must be true.

According to Bochvar, such sentences are neither true of false but rather **paradoxical** or **meaningless**

B Semantics

Bochvar's semantics follows the principle that the third logical value, denoted now by m (for miningless) is in some sense "infectious";

if one component of the formula is **assigned** the value m then the formula is also **assigned** the value m

Bochvar also adds an one **assertion** operator S that **asserts** the logical value of T and F , i.e.

$$SF = F, \quad ST = T$$

S also **asserts** that meaningfulness m is false, i.e

$$Sm = F$$

B Semantics: Language

Language: we add a new **one argument** connective S and get

$$\mathcal{L}_B = \mathcal{L}_{\{\neg, S, \Rightarrow, \cup, \cap\}}$$

We denote by $\mathcal{F}_B$ the set of all formulas of the language $\mathcal{L}_B$ and by $\mathcal{F}$ the set of formulas of the language $\mathcal{L}_{\{\neg, \Rightarrow, \cup, \cap\}}$ common to the classical and all 3 valued logics considered till now.

Observe that directly from the definition we have that

$$\mathcal{F} \subset \mathcal{F}_{\mathcal{B}}$$

The formula SA **reads** "assert A"

B Semantics: Connectives

Negation

$\neg$	F	m	T
	T	m	F

Conjunction

$\cap$	F	m	T
F	F	m	F
m	m	m	m
T	F	m	T

B Semantics: Connectives

Disjunction

$\cup$	F	m	T
F	F	m	T
m	m	m	m
T	T	m	T

Implication

$\Rightarrow$	F	m	T
F	T	m	T
m	m	m	m
T	F	m	T

B Semantics: Connectives, Tautology

Assertion

S	F	m	T
	F	F	T

For all other steps of **definition** of **B** semantics we follow the standard established for the **M** semantics, as we did in all previous cases

In particular the set of all **B** **tautologies** is

$$\mathbf{BT} = \{A \in \mathcal{F} : \models_{\mathbf{B}} A\}$$

B Semantics: Tautology

We get by easy evaluation that

$$\models_{\mathbf{B}} (Sa \cup \neg Sa)$$

This proves that $\mathbf{BT} \neq \emptyset$, what means that

B semantics is **well defined**

B Semantics: Tautology

Observe that **not all** formulas containing the connective S are **B** **tautologies**, for example we have that

$$\not\models_{\mathbf{B}} (a \cup \neg Sa), \quad \not\models_{\mathbf{B}} (Sa \cup \neg a), \quad \not\models_{\mathbf{B}} (Sa \cup S\neg a)$$

as any truth assignment **v** such that

$$v(a) = m$$

is a **counter model** for all of them, because

$m \cup x = m$ for all $x \in \{F, m, T\}$ and

$Sm \cup S\neg m = F \cup Sm = F \cup F = F$

B Semantics: Tautology

Let A be a formula that **do not** contain the assertion operator S, i.e. the formula $A \in \mathcal{F}$ of the language $\mathcal{L}_{\{\neg, \Rightarrow, \cup, \cap\}}$

Any v, such that v(a) = m for at least one variable in the formula $A \in \mathcal{F}$ is a **counter-model** for that formula, i.e.

$$\mathbf{T} \cap \mathbf{BT} = \emptyset$$

Observation

A formula $A \in \mathcal{F}_B$ to be **considered** to be a **B** tautology must contain the connective S in front of **each** variable appearing in A