cse371/mat371
LOGIC

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LECTURE 3b

Chapter 3
Propositional Semantics: Classical and Many Valued

Extensional Semantics M

Extensional Semantics **M** - Introduction

Given a propositional language $\mathcal{L}_{CON}$, the symbols for its connectives always have some intuitive **meaning**

A formal **definition** of the meaning of these symbols is called a **semantics** for the language $\mathcal{L}_{CON}$

A given language $\mathcal{L}_{CON}$ can have different semantics but we always **define** them in order to single out special formulas of the language, called **tautologies**

Tautologies are formulas of the language that are always true under a given **semantics**

Extensional Semantics **M** Introduction

We have already introduced the intuitive and formal notions of a classical **semantics**, discussed its motivation and underlying assumptions

The **classical semantics** assumption is that it considers only two logical values. The other one is that all classical propositional connectives are **extensional**

We have also observed that in everyday language there are expressions such as "I believe that", "it is possible that", " certainly", etc .... and that they are represented by some propositional connectives which are **not extensional**

Extensional Semantics **M** Introduction

Non-extensional connectives **do not** play any role in mathematics and so **are not** discussed in classical logic and will be studied separately

The **extensional connectives** are defined intuitively as such that the logical value of the formulas form by means of these **connectives** and certain given formulas **depends only** on the logical value(s) of the given formulas

Extensional Connectives Definition

We adopt a following **formal** definition of extensional connectives for a propositional language $\mathcal{L}_{CON}$

Definition

Let $\mathcal{L}_{CON}$ be such that $CON = C_1 \cup C_2$, where C_1, C_2 are the sets of unary and binary connectives, respectively

Let LV be a non-empty set of **logical values**

A connective $\triangledown \in C_1$ or $\circ \in C_2$ is called **extensional** if it is defined by a respective function

$$\triangledown : \ LV \longrightarrow LV \quad \text{or} \quad \circ : \ LV \times LV \longrightarrow LV$$

Extensional Semantics **M** Introduction

A semantics **M** for a language $\mathcal{L}_{CON}$ is called **extensional** provided all connectives in CON are extensional and its notion of **tautology** is defined terms of the connectives and their logical values

A semantics with a set of m logical values is called a m-valued **extensional**

The **classical** semantics is a special case of a 2-valued **extensional** semantics

Classical semantics **defines** classical logic with its set of classical propositional **tautologies**

Many of logics are defined by various **extensional semantics** with sets of logical values LV with more then 2 elements

Extensional Semantics **M** Introduction

The languages of many important **logics** like modal, multi-modal, knowledge, believe, temporal, contain **connectives** that are **not extensional** because they are defined by **non-extensional** semantics

The intuitionistic logic is based on the **same** language as the classical one and its Kripke Models semantics is **not extensional**

Defining a **semantics** for a given language means **more** then defining connectives

The ultimate **goal** of any semantics is to define the notion of its own **tautology**

Extensional Semantics **M** Introduction

In order to **define** which formulas of a given

$$\mathcal{L}_{CON}$$

we want to to be **tautologies** under a given semantics **M** we **assume** that the set LV of logical values of **M** always has a **distinguished** logical value, often denoted by T for "absolute" truth

We also can **distinguish**, and often we do, another special value F representing "absolute" falsehood

We will use these symbols T, F for "absolute" truth and falsehood

We may also use other symbols like 1, 0 or others

Extensional Semantics **M** Introduction

The ”absolute” truth value T serves to **define** a notion of a **tautology** (as a formula always ”true”)

Extensional semantics share not only the similar pattern of **defining** their (extensional) connectives, but also the method of **defining** the notion of a **tautology**

We hence **define** a general notion of an extensional semantics as sequence of **steps** leading to the definition of a **tautology**

Extensional Semantics **M** Introduction

Here are the **steps** leading to the definition of a **tautology**

Step 1 We **define** all extensional connectives of **M**

Step 2 We **define** main component of the definition of a **tautology**, namely a **function** v that assigns to any formula $A \in \mathcal{F}$ its logical **value** from LV

The function v is often called a **truth assignment** and we will use this name

Extensional Semantics **M** Introduction

Step 3 Given a truth assignment v and a formula $A \in \mathcal{F}$, we **define** what does it mean that

v **satisfies** A

i.e. we define a notion saying that v is a **model** for A under semantics **M**

Step 4 We **define** a notion of tautology as follows

A is a **tautology** under semantics **M** if and only if **all** truth assignments v **satisfy** A

i.e. that **all** truth assignments v are **models** for A

We use a notion of a **model** because it is an important, if not the most important notion of modern **logic**

The notion of a **model** is usually defined in terms of the notion of **satisfaction**

In **classical** propositional logic these notions are the same and the **use** of expressions

" v **satisfies** A" and "v is a **model** for A"

is **interchangeable**

This also is true for of any propositional **extensional semantics** and in particular it holds for m-valued semantics discussed later in this chapter

Extensional Semantics **M** Introduction

The notions of **satisfaction** and **model** are not interchangeable for **predicate** languages semantics

We already discussed intuitively the notion of **model** and **satisfaction** for predicate language in chapter 2

We will define them in **full formality** in chapter 8

The use of the notion of a **model** also allows us to adopt and discuss the standard predicate logic **definitions** of consistency and independence for **propositional** case

Extensional Semantics **M** Formal Definition

Definition

Any formal definition of an **extensional semantics** **M** for a given language $\mathcal{L}_{CON}$ consists of **specifying** the following steps defining its main components

Step 1 We define a set LV of logical values, its distinguished value T, and define all connectives of $\mathcal{L}_{CON}$ to be **extensional**

Step 2 We define notion of a truth assignment and its extension

Step 3 We define notions of satisfaction, model, counter model

Step 4 We define notion of a **tautology** under the semantics **M**

Extensional Semantics **M** Formal Definition

What differs one semantics from the other is the **choice** of the set LV of logical values and **definition** of the connectives of $\mathcal{L}_{CON}$, that are defined in the first step below

Step 1 We adopt a following formal definition of **extensional** connectives of $\mathcal{L}_{CON}$

Definition

Let $\mathcal{L}_{CON}$ be such that $CON = C_1 \cup C_2$, where C_1, C_2 are the sets of unary and binary connectives, respectively

Let LV be a non-empty set of **logical values**

A connective $\triangledown \in C_1$ or $\circ \in C_2$ is called **extensional** if it is defined by a respective function

$$\triangledown : LV \longrightarrow LV \quad \text{or} \quad \circ : LV \times LV \longrightarrow LV$$

M Truth Assignment Formal Definition

Step 2 We define a function called **truth assignment** and its **extension** in terms of the propositional connectives as defined in the **Step 1**

Definition

Let LV be the set of logical values of **M** and VAR the set of propositional variables of the language $\mathcal{L}_{CON}$

Any function

$$v : \ VAR \longrightarrow LV$$

is called a **truth assignment** under semantics **M**

We call it for short a **M truth assignment**

We use the term **M** truth assignment and **M** truth extension to stress that it is defined relatively to a given semantics **M**

M Truth Extension Formal Definition

Definition

Given a **M** truth assignment $v : VAR \longrightarrow LV$

We define its **extension** v^* to the set $\mathcal{F}$ of all formulas of $\mathcal{L}_{CON}$ as any function

$$v^* : \mathcal{F} \longrightarrow LV$$

such that the following conditions are satisfied.

(i) for any $a \in VAR$,

$$v^*(a) = v(a);$$

(ii) For any connectives $\nabla \in C_1$, $\circ \in C_2$, and for any formulas $A, B \in \mathcal{F}$,

$$v^*(\nabla A) = \nabla v^*(A) \quad \text{and} \quad v^*((A \circ B)) = \circ(v^*(A), v^*(B))$$

We call the v^* the **M truth extension**

M Truth Extension Formal Definition

Remark

The symbols on the **left-hand side** of the equations

$$v^*(\triangledown A) = \triangledown v^*(A) \quad \text{and} \quad v^*((A \circ B)) = \circ(v^*(A), v^*(B))$$

represent connectives in their natural language meaning and the symbols on the **right-hand side** represent connectives in their semantical meaning as defined in the **Step1**

M Truth Extension Formal Definition

We use names " **M truth assignment**" and " **M truth extension**" to stress that we define them for the set of logical values of the semantics **M**

Notation Remark

For any function g, we use a symbol g^* to denote its **extension** to a larger domain

Mathematician often use the same symbol g for both a function g and its extension g^*

Step 3 The notions of **satisfaction** and **model** are interchangeable in **M** semantics and we define them as follows.

Definition

Given an **M** truth assignment $v : VAR \longrightarrow LV$ and its **M** truth extension v^* Let $T \in LV$ be the distinguished logical truth value

We say that the truth assignment v **M satisfies** a formula A if and only if $v^*(A) = T$

We write symbolically

$$v \models_{\mathbf{M}} A$$

Any truth assignment v, such that $v \models_{\mathbf{M}} A$ is called an **M model** for the formula A

Counter Model

Definition

Given an **M** truth assignment $v : VAR \longrightarrow LV$ and its **M** truth extension v^*. Let $T \in LV$ be the distinguished logical truth value

We say that the truth assignment v **M does not satisfy** a formula A if and only if $v^*(A) \neq T$

We write symbolically

$$v \not\models_{\mathbf{M}} A$$

Any truth assignment v, such that $v \not\models_{\mathbf{M}} A$ is called an **M counter model** for the formula A

M Tautology

Step 4 We define the notion of **M tautology** as follows

Definition

A formula A is an **M tautology** if and only if

$v \models_{\mathbf{M}} A$, for all truth assignments $v : VAR \longrightarrow LV$

We denote it as

$$\models_{\mathbf{M}} A$$

We also say that

A is an **M tautology** if and only if all truth assignments $v : VAR \longrightarrow LV$ are **M models** for A

M Tautology

Observe that directly from definition of the **M model** we get the following equivalent form of the definition of **tautology**

Definition

A formula A is an **M tautology** if and only if
$v^*(A) = T$, for all truth assignments $v : VAR \longrightarrow LV$

We denote by **MT** the set of **all tautologies** under the semantic **M**, i.e.

$$\mathbf{MT} = \{A \in \mathcal{F} : \models_{\mathbf{M}} A\}$$

M Tautology

Obviously, when we **develop a logic** by defining its semantics we want the semantics to be such that the **logic** has a non empty set of its tautologies

We **express** it in a form of the following definition

Definition

Given a language $\mathcal{L}_{CON}$ and its extensional semantics **M**

The semantics **M** is **well defined** if and only if its set **MT** of all tautologies is non empty, i.e. when

$$\mathbf{MT} \neq \emptyset$$

Extensional Semantics **M**

As the next steps we use the **definitions** established here to define and discuss in details the following particular cases of the extensional semantics **M**

Many valued **semantics** have their beginning in the work of Łukasiewicz (1920)

He was the first to define a 3- valued extensional semantics for a language $\mathcal{L}_{\{\neg,\cap,\cup,\Rightarrow\}}$ of classical logic, and called it a 3- valued logic, for short

Extensional Semantics **M**

The other logics defined by various **extensional semantics** followed and we will discuss some of them

In particular we present Heyting's 3-valued **semantics** as an introduction to the discussion of **first** ever semantics for the intuitionistic logic and some modal logics

Challenge Exercise

1. Define your own propositional language $\mathcal{L}_{CON}$ that contains also **different connectives** that the standard connectives $\neg, \cup, \cap, \Rightarrow$

Your language $\mathcal{L}_{CON}$ **does not need** to include all (if any!) of the standard connectives $\neg, \cup, \cap, \Rightarrow$

2. Describe intuitive meaning of the new connectives of your language

3. Give some motivation for **your own** semantic **M**

4. Define formally your own extensional semantics **M** for your language $\mathcal{L}_{CON}$

Write carefully all **Steps 1- 4** of the definition of your **M**