CSE/MAT371 QUIZ 1 SOLUTIONS Fall 2018

QUESTION 1

Write the following natural language statement:

From the fact that each natural number is greater than zero we deduce that it is not possible that Anne is a boy or, if it is possible that Anne is not a boy, then it is necessary that it is not true that each natural number is greater than zero

in the following two ways

1. As a formula $A_1 \in \mathcal{F}_1$ of a language $\mathcal{L}_{\{\neg,\ \Box,\ \Diamond,\ \cap,\ \cup,\ \Rightarrow\}}$, **2.** As a formula $A_2 \in \mathcal{F}_2$ of a language $\mathcal{L}_{\{\neg,\ \cap,\ \cup,\ \Rightarrow\}}$

Solution of 1.

Propositional Variables: a, b
a denotes statement: *each natural number is greater than zero* ,
b denotes a statement: *Anne is a boy*

Formula is $(a \Rightarrow (\neg\Diamond b \cup (\Diamond\neg b \Rightarrow \Box\neg a)))$

Solution of 2.

Propositional Variables: a, b, c, d
a denotes statement: *each natural number is greater than zero* ,
b denotes a statement: *it is not possible that Anne is a boy* ,
c denotes a statement: *it is possible that Anne is not a boy*, and
d denotes a statement: *it is necessary that it is not true that each natural number is greater than zero*

Formula is $(a \Rightarrow (\neg b \cup (c \Rightarrow d)))$

QUESTION 2

Circle formulas that **are** propositional/ predicate **tautologies**

1. $\mathcal{S}_1 = \{\, (A \Rightarrow (A \cup \neg B)),\ \ (((a \Rightarrow b) \cap (a \Rightarrow c)) \Rightarrow (a \Rightarrow b)),\ \ ((a \Rightarrow b) \cup (a \cap \neg b)),\ \ (A \cup (A \Rightarrow B)),\ \ (a \cup \neg b) \,\}$

Solution

$\not\models (a \cup \neg b)$, all other formulas are tautologies

2. $\mathcal{S}_2 = \{\, (\forall x\, A(x) \Rightarrow \exists x\, A(x)),\ \ (\forall x\, (\neg P(x,y) \cup P(x,y)) \Rightarrow \exists x\, P(x,y)),\ \ ((\exists x A(x) \cap \exists x B(x)) \Rightarrow \exists x\, (A(x) \cap B(x))) \,\}$

Solution

$\models (\forall x\, A(x) \Rightarrow \exists x\, A(x))$, all other formulas are NOT tautologies

QUESTION 3

Here is a mathematical statement **S**:

For all real numbers x the following holds: If $x < 0$, *then there is a natural number n, such that* $x + n < 0$."

1. Re-write **S** as a symbolic mathematical statement **SF** that only uses mathematical and logical symbols.

2. Translate the symbolic statement **SF** into to a corresponding formula with **restricted quantifiers**

3. Translate your **restricted domain** quantifiers logical formula into a correct formula A of the predicate language $\mathcal{L}$

Solution

The statement **S** is:

"For all real numbers x the following holds: If $x < 0$, *then there is a natural number n, such that* $x + n < 0$."

S becomes a symbolic mathematical statement **SF** $\forall_{x \in R}(x < 0 \Rightarrow \exists_{n \in N}\ x + n < 0)$

We write R(x) for $x \in R$, N(y) for $n \in N$, a constant c for the number 0

We use $L \in \mathbf{P}$ to denote the relation <, we use $f \in \mathbf{F}$ to denote the function +.

The statement $x < 0$ becomes an **atomic formula** L(x, c).

The statement $x + n < 0$ becomes an **atomic formula** L(f(x,y), c).

The symbolic mathematical statement **SF** $\forall_{x \in R}(x < 0 \Rightarrow \exists_{n \in N}\ x + n < 0)$

becomes a **restricted quantifiers** formula $\forall_{R(x)}(L(x, c) \Rightarrow \exists_{N(y)} L(f(x, y), c))$.

We apply now the **transformation rules** and get a corresponding formula $A \in \mathcal{F}$:

$\forall x(N(x) \Rightarrow (L(x, c) \Rightarrow \exists y(N(y) \cap L(f(x, y), c)))$.

QUESTION 4

Given a formula A : $\forall x \exists y\ P(f(x, y), c)$ of the predicate language $\mathcal{L}$, and

two **model structures** $\mathbf{M_1} = (Z, I_1)$, $\mathbf{M_2} = (N, I_2)$ with the interpretations defined as follows.

$P_{I_1} : =, \quad f_{I_1} : +, \quad c_{I_1} : 0$ and $P_{I_2} : >, \quad f_{I_2} : \cdot, \quad c_{I_2} : 0$.

1. Show that $\mathbf{M_1} \models A$

Solution

A_{I_1} : $\forall_{x \in Z} \exists_{y \in Z}\ x + y = 0$ is a **true** statement;

For each $x \in Z$ exists $y = -x$ and $-x \in Z$ and $x - x = 0$.

2. Show that $\mathbf{M_2} \not\models A$.

Solution

A_{I_2} : $\forall_{x \in N} \exists_{y \in N}\ x \cdot y > 0$ is a **false** statement for $x = 0$.

QUESTION 5

Given a formula $A = (\neg a \Rightarrow (\neg b \cup (b \Rightarrow \neg c)))$

and its **restricted model** $v_A : \{a, b, c\} \longrightarrow \{T, F\}$, $v_A(a) = T,\ v_A(b) = T,\ v_A(c) = F$

Extend v_A to the set of all propositional variables VAR to obtain 2 different, non restricted **models** for A

Solution

Model w_1 is a function

$w_1 : VAR \longrightarrow \{T, F\}$ such that

$w_1(a) = v_A(a) = T,\ \ w_1(b) = v_A(b) = T,$

$w_1(c) = v_A(c) = F,$ and $w_1(x) = T,$ for all $x \in VAR - \{a, b, c\}$

Model w_2 is defined by a formula

$w_2(a) = v_A(a) = T,\ \ w_2(b) = v_A(b) = T,$

$w_2(c) = v_A(c) = F,$ and $w_2(x) = F,$ for all $x \in VAR - \{a, b, c\}$

REMINDER: we define H semantics operations $\cup$ and $\cap$ as $x \cup y = max\{x,\ y\}$, $x \cap y = min\{x,\ y\}$

the implication and negation are defined as

$$x \Rightarrow y = \begin{cases} T & \text{if } x \leq y \\ y & \text{otherwise} \end{cases}$$

$$\neg x = x \Rightarrow F$$

QUESTION 6

We know that $v : VAR \longrightarrow \{F, \perp, T\}$ is such that $v^*((a \cap \neg b) \Rightarrow (a \Rightarrow c)) = \perp$ under H semantics.

Evaluate $v^*(((b \Rightarrow a) \Rightarrow b))$. You can use SHORTHAND notation.

Solution

The Truth Tables for Implication and Negation are:

H-Implication

$\Rightarrow$	F	$\perp$	T
F	T	T	T
$\perp$	F	T	T
T	F	$\perp$	T

H Negation

$\neg$	F	$\perp$	T
	T	F	F

We KNOW that

$v^*((a \cap \neg b) \Rightarrow (a \Rightarrow c)) = \perp$ under **H** semantics if and only if (we use shorthand notation)

iff $(a \cap \neg b) = T$ and $(a \Rightarrow c) = \perp$

iff $a = T,\ b = F,$ and $(T \Rightarrow c) = \perp$

iff $a = T,\ b = F, c = \perp$

Hence we evaluate

$v^*(((b \Rightarrow a) \Rightarrow b)) = (F \Rightarrow T) \Rightarrow F) = T \Rightarrow F = F$