cse371/mat371
LOGIC

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LECTURE 10a

Chapter 6
Automated Proof Systems for Classical Propositional Logic

PART 3: GENTZEN SYSTEMS

Gentzen Sequent Calculus GL

The proof system **GL** for the classical propositional logic presented now is a version of the original **Gentzen** (1934) systems **LK**.

A **constructive** proof of the Completeness Theorem for the system **GL** is very similar to the proof of the Completeness Theorem for the system **RS**

Expressions of the system are like in the original Gentzen system **LK** are Gentzen **sequents**

Hence we use also a name Gentzen sequent calculus for it

Gentzen Sequent Calculus GL

Language of **GL**

$$\mathcal{L} = \mathcal{L}_{\{\cup,\cap,\Rightarrow,\neg\}}$$

Let $\mathcal{F}$ be the set of all formulas of $\mathcal{L}_{\{\cup,\cap,\Rightarrow,\neg\}}$

We **denote**, as in the **RS** system, the finite sequences (with indices if necessary) of formulas by Greek capital letters

$$\Gamma, \Delta, \Sigma, \ldots$$

We **add** a new symbol $\longrightarrow$ called a **Gentzen arrow** to the language $\mathcal{L}$, i.e. we form a language

$$\mathcal{L}_1 = \mathcal{L} \cup \{\longrightarrow\}.$$

The Gentzen **sequents** are built out of finite sequences (empty included) of formulas, i.e. elements of $\mathcal{F}^*$, and the additional symbol $\longrightarrow$

Gentzen Sequents

Definition Any expression

$$\Gamma \longrightarrow \Delta$$

where $\Gamma, \Delta \in \mathcal{F}^*$ is called a sequent

Intuitively, we interpret **semantically** a sequent

$$A_1, ..., A_n \longrightarrow B_1, ..., B_m$$

where $n, m \geq 1$, as a formula

$$(A_1 \cap ... \cap A_n) \Rightarrow (B_1 \cup ... \cup B_m)$$

of the language $\mathcal{L}_{\{\cup, \cap, \Rightarrow, \neg\}}$

Gentzen Sequents

The sequent

$$A_1, ..., A_n \longrightarrow$$

(where $m \geq 1$) means that $A_1 \cap ... \cap A_n$ yields a **contradiction**

The sequent

$$\longrightarrow B_1, ..., B_m$$

(where $m \geq 1$) means semantically $T \Rightarrow (B_1 \cup ... \cup B_m)$

The empty sequent

$$\longrightarrow$$

means a **contradiction**

Gentzen Sequents

Given non empty sequences Γ, Δ

We denote by σ_Γ any conjunction of all formulas of Γ

We denote by δ_Δ any disjunction of all formulas of Δ

The **intuitive semantics** of a non- empty sequent $\Gamma \longrightarrow \Delta$ is

$$\Gamma \longrightarrow \Delta \equiv (\sigma_\Gamma \Rightarrow \delta_\Delta)$$

Formal Semantics

Formal semantics for sequents of **GL** is defined as follows

Let $v : VAR \longrightarrow \{T, F\}$ be a truth assignment and v^* its extension to the set of formulas $\mathcal{F}$ of $\mathcal{L}_{\{\cup,\cap,\Rightarrow,\neg\}}$

We **extend** v^* to the set

$$SQ = \{\, \Gamma \longrightarrow \Delta : \quad \Gamma, \Delta \in \mathcal{F}^* \,\}$$

of all sequents as follows

For any sequent $\Gamma \longrightarrow \Delta \in SQ$

$$v^*(\Gamma \longrightarrow \Delta) \;=\; v^*(\sigma_\Gamma) \Rightarrow v^*(\delta_\Delta)$$

Formal Semantics

In the case when $\Gamma = \emptyset$ or $\Delta = \emptyset$ we **define**

$$v^*(\longrightarrow \Delta) = (T \Rightarrow v^*(\delta_\Delta))$$

$$v^*(\Gamma \longrightarrow) = (v^*(\sigma_\Gamma) \Rightarrow F)$$

The sequent $\Gamma \longrightarrow \Delta$ is **satisfiable** if there is a truth assignment $v : VAR \longrightarrow \{T, F\}$ such that

$$v^*(\Gamma \longrightarrow \Delta) = T$$

Formal Semantics

Model for $\Gamma \longrightarrow \Delta$ is any v such that

$$v^*(\Gamma \longrightarrow \Delta) = T$$

We write it $v \models \Gamma \longrightarrow \Delta$

Counter- model is any v such that

$$v^*(\Gamma \longrightarrow \Delta) = F$$

We write it $v \not\models \Gamma \longrightarrow \Delta$

Tautology is any sequent $\Gamma \longrightarrow \Delta$ such that

$v^*(\Gamma \longrightarrow \Delta) = T$ for all truth assignments $v : VAR \longrightarrow \{T, F\}$

We write it

$$\models \Gamma \longrightarrow \Delta$$

Example

Let $\Gamma \longrightarrow \Delta$ be a sequent

$$a, (b \cap a) \longrightarrow \neg b, (b \Rightarrow a)$$

The truth assignment v for which

$$v(a) = T \quad \text{and} \quad v(b) = T$$

is a **model** for $\Gamma \longrightarrow \Delta$ as shows the following computation

$$v^*(a, (b \cap a) \longrightarrow \neg b, (b \Rightarrow a)) = v^*(\sigma_{\{a,(b \cap a)\}}) \Rightarrow v^*(\delta_{\{\neg b,(b \Rightarrow a)\}})$$

$$= v(a) \cap (v(b) \cap v(a)) \Rightarrow \neg v(b) \cup (v(b) \Rightarrow v(a))$$

$$= T \cap T \cap T \Rightarrow \neg T \cup (T \Rightarrow T) = T \Rightarrow (F \cup T) = T \Rightarrow T = T$$

Example

Observe that the truth assignment v for which

$$v(a) = T \quad \text{and} \quad v(b) = T$$

is the **only one** for which

$$v^*(\Gamma) = v^*(a, (b \cap a) = T$$

and we proved that it is a **model** for

$$a, (b \cap a) \longrightarrow \neg b, (b \Rightarrow a)$$

It is hence **impossible** to find v which would **falsify it**, what proves that

$$\models a, (b \cap a) \longrightarrow \neg b, (b \Rightarrow a)$$

Gentzen System **GL**

Definition of **GL**

Logical Axioms LA

We adopt as an axiom any sequent of variables (positive literals) which contains a propositional variable that appears on both sides of the sequent arrow $\longrightarrow$, i.e any sequent of the form

$$\Gamma'_1, a, \Gamma'_2 \longrightarrow \Delta'_1, a, \Delta'_2$$

for any $a \in VAR$ and any sequences $\Gamma'_1, \Gamma'_2, \Delta'_1, \Delta'_2 \in VAR^*$

Gentzen System **GL**

Inference rules of **GL**

Let $\Gamma^{'}, \Delta^{'} \in VAR^{*}$ and $\Gamma, \Delta \in \mathcal{F}^{*}$

Conjunction rules

$$(\cap \rightarrow) \quad \frac{\Gamma^{'},\ A, B,\ \Gamma\ \longrightarrow\ \Delta^{'}}{\Gamma^{'},\ (A \cap B),\ \Gamma\ \longrightarrow\ \Delta^{'}}$$

$$(\rightarrow \cap) \quad \frac{\Gamma\ \longrightarrow\ \Delta,\ A,\ \Delta^{'}\ \ ;\ \ \Gamma\ \longrightarrow\ \Delta,\ B,\ \Delta^{'}}{\Gamma\ \longrightarrow\ \Delta,\ (A \cap B)\ \Delta^{'}}$$

Gentzen System **GL**

Disjunction rules

$$(\longrightarrow \cup) \quad \frac{\Gamma \longrightarrow \Delta,\ A, B,\ \Delta'}{\Gamma \longrightarrow \Delta,\ (A \cup B),\ \Delta'}$$

$$(\cup \longrightarrow) \quad \frac{\Gamma',\ A,\ \Gamma \longrightarrow \Delta' \quad ; \quad \Gamma',\ B,\ \Gamma \longrightarrow \Delta'}{\Gamma',\ (A \cup B),\ \Gamma \longrightarrow \Delta'}$$

Gentzen System **GL**

Implication rules

$$(\rightarrow\Rightarrow)\quad \frac{\Gamma',\ A,\ \Gamma\ \longrightarrow\ \Delta,\ B,\ \Delta'}{\Gamma',\Gamma\ \longrightarrow\ \Delta,\ (A\Rightarrow B),\ \Delta'}$$

$$(\Rightarrow\rightarrow)\quad \frac{\Gamma',\Gamma\ \longrightarrow\ \Delta,\ A,\ \Delta' \quad ; \quad \Gamma',\ B,\ \Gamma\ \longrightarrow\ \Delta,\Delta'}{\Gamma',\ (A\Rightarrow B),\ \Gamma\ \longrightarrow\ \Delta,\Delta'}$$

Gentzen System **GL**

Negation rules

$$(\neg \rightarrow) \quad \frac{\Gamma', \Gamma \longrightarrow \Delta, A, \Delta'}{\Gamma', \neg A, \Gamma \longrightarrow \Delta, \Delta'}$$

$$(\rightarrow \neg) \quad \frac{\Gamma', A, \Gamma \longrightarrow \Delta, \Delta'}{\Gamma', \Gamma \longrightarrow \Delta, \neg A, \Delta'}$$

Gentzen System **GL**

We define the Gentzen System **GL**

$$\mathbf{GL} = (\ \mathcal{L}_{\{\cup,\cap,\Rightarrow,\neg\}},\ \ SQ,\ \ LA,\ \ \mathcal{R}\ )$$

for

$$\mathcal{R} = \{(\cap \longrightarrow),\ (\longrightarrow \cap),\ (\cup \longrightarrow),\ (\longrightarrow \cup),\ (\Rightarrow \longrightarrow),\ (\longrightarrow \Rightarrow)\}$$

$$\cup \{(\neg \longrightarrow),\ (\longrightarrow \neg)\}$$

We write, as usual,

$$\vdash_{\mathbf{GL}} \Gamma \longrightarrow \Delta$$

to denote that $\Gamma \longrightarrow \Delta$ has a formal proof in **GL**

A formula $A \in \mathcal{F}$, has a proof in **GL** if the sequent $\longrightarrow A$ has a proof in **GL**, i.e.

$$\vdash_{\mathbf{GL}} A \quad \text{if ad only if} \quad \longrightarrow A$$

Gentzen System **GL**

We consider, as we did with **RS** the proof trees for **GL**, i.e. we define

A **proof tree**, or **GL**-proof of $\Gamma \longrightarrow \Delta$ is a tree

$$\mathbf{T}_{\Gamma \longrightarrow \Delta}$$

of sequents satisfying the following conditions:

1. The topmost sequent, i.e **the root** of $\mathbf{T}_{\Gamma \longrightarrow \Delta}$ is $\Gamma \longrightarrow \Delta$

2. All **leafs** are axioms

3. The **nodes** are sequents such that each sequent on the tree follows from the ones immediately preceding it by one of the rules.

Gentzen System **GL**

Remark

The proof search in **GL** as defined by the decomposition tree for a given formula A **is not always unique**

We show it on an example on the next slide

Example

A tree-proof in **GL** of the de Morgan Law

$$\longrightarrow (\neg(a \cap b) \Rightarrow (\neg a \cup \neg b))$$

$$\mid (\longrightarrow \Rightarrow)$$

$$\neg(a \cap b) \longrightarrow (\neg a \cup \neg b)$$

$$\mid (\longrightarrow \cup)$$

$$\neg(a \cap b) \longrightarrow \neg a, \neg b$$

$$\mid (\longrightarrow \neg)$$

$$b, \neg(a \cap b) \longrightarrow \neg a$$

$$\mid (\longrightarrow \neg)$$

$$b, a, \neg(a \cap b) \longrightarrow$$

$$\mid (\neg \longrightarrow)$$

$$b, a \longrightarrow (a \cap b)$$

$$\bigwedge (\longrightarrow \cap)$$

$$b, a \longrightarrow a \qquad\qquad b, a \longrightarrow b$$

Example

Here is another tree-proof in **GL** of the de Morgan Law

$$\longrightarrow \; (\neg(a \cap b) \Rightarrow (\neg a \cup \neg b))$$

$$\mid (\longrightarrow \Rightarrow)$$

$$\neg(a \cap b) \; \longrightarrow \; (\neg a \cup \neg b)$$

$$\mid (\longrightarrow \cup)$$

$$\neg(a \cap b) \; \longrightarrow \; \neg a, \neg b$$

$$\mid (\longrightarrow \neg)$$

$$b, \neg(a \cap b) \; \longrightarrow \; \neg a$$

$$\mid (\neg \longrightarrow)$$

$$b \; \longrightarrow \; \neg a, (a \cap b)$$

$$\bigwedge (\longrightarrow \cap)$$

$$b \longrightarrow \; \neg a, a \qquad\qquad b \longrightarrow \; \neg a, b$$

$$\mid (\longrightarrow \neg) \qquad\qquad \mid (\longrightarrow \neg)$$

$$b, a \longrightarrow \; a \qquad\qquad b, a \longrightarrow \; b$$

Gentzen System **GL**

The process of **searching for proofs** of a formula A in **GL** consists, as in the **RS** type systems, of building certain trees, called decomposition trees

Their construction is similar to the one for **RS** type systems

We take a **root** of a **decomposition tree** T_A of of a formula A a sequent $\longrightarrow A$

For each **node**, if there is a rule of **GL** which conclusion has the same form as **node** sequent, then the **node** has **children** that are **premises** of the rule

If the **node** consists only of a sequent built only out of variables then it **does not** have any children

This is a termination condition for the tree

Gentzen System **GL**

We **prove** that each formula A generates a finite set of decomposition trees, $\mathcal{T}_A$, such that the following holds

If there exist a tree $T_A \in \mathcal{T}_A$ whose **all leaves** are axioms, then tree T_A constitutes a **proof** of A in **GL**

If **all trees** in $\mathcal{T}_A$ have at **least one** non-axiom leaf, the proof of A **does not exist**

The first step in **defining** a notion of a decomposition tree consists of transforming the inference rules of **GL**, as we did in the case of the **RS** type systems, into corresponding **decomposition rules**

Decomposition Rules of **GL**

Decomposition rules of **GL**

Let $\Gamma', \Delta' \in VAR^*$ and $\Gamma, \Delta \in \mathcal{F}^*$

Conjunction rules

$$(\cap \rightarrow) \quad \frac{\Gamma', (A \cap B), \Gamma \longrightarrow \Delta'}{\Gamma', A, B, \Gamma \longrightarrow \Delta'}$$

$$(\rightarrow \cap) \quad \frac{\Gamma \longrightarrow \Delta, (A \cap B)\, \Delta'}{\Gamma \longrightarrow \Delta, A, \Delta' \quad ; \quad \Gamma \longrightarrow \Delta, B, \Delta'}$$

Decomposition Rules of **GL**

Disjunction rules

$$(\longrightarrow \cup) \quad \frac{\Gamma \longrightarrow \Delta, (A \cup B), \Delta'}{\Gamma \longrightarrow \Delta, A, B, \Delta'}$$

$$(\cup \longrightarrow) \quad \frac{\Gamma', (A \cup B), \Gamma \longrightarrow \Delta'}{\Gamma', A, \Gamma \longrightarrow \Delta' \quad ; \quad \Gamma', B, \Gamma \longrightarrow \Delta'}$$

Decomposition Rules of **GL**

Implication rules

$$(\rightarrow\Rightarrow)\quad \frac{\Gamma^{'}, \Gamma \ \longrightarrow\ \Delta,\ (A \Rightarrow B),\ \Delta^{'}}{\Gamma',\ A,\ \Gamma \ \longrightarrow\ \Delta,\ B,\ \Delta^{'}}$$

$$(\Rightarrow\rightarrow)\quad \frac{\Gamma^{'},\ (A \Rightarrow B),\ \Gamma \ \longrightarrow\ \Delta, \Delta^{'}}{\Gamma', \Gamma \ \longrightarrow\ \Delta,\ A,\ \Delta' \quad ; \quad \Gamma',\ B,\ \Gamma \ \longrightarrow\ \Delta, \Delta'}$$

Decomposition Rules of **GL**

Negation rules

$$(\neg \rightarrow) \quad \frac{\Gamma', \neg A, \Gamma \longrightarrow \Delta, \Delta'}{\Gamma', \Gamma \longrightarrow \Delta, A, \Delta'}$$

$$(\rightarrow \neg) \quad \frac{\Gamma', \Gamma \longrightarrow \Delta, \neg A, \Delta'}{\Gamma', A, \Gamma \longrightarrow \Delta, \Delta'}$$

Decomposition Tree Definition

For each formula $A \in \mathcal{F}$, a decomposition tree $\mathbf{T}_A$ is a tree build as follows

Step 1. The sequent $\longrightarrow A$ is the **root** of $\mathbf{T}_A$

For any node $\Gamma \longrightarrow \Delta$ of the tree we follow the steps below

Step 2. If $\Gamma \longrightarrow \Delta$ is **indecomposable**, then $\Gamma \longrightarrow \Delta$ becomes a **leaf** of the tree

Step 3. If $\Gamma \longrightarrow \Delta$ is **decomposable**, then we pick a decomposition rule that **matches** the sequent of the current node

To do so we **proceed** as follows

Decomposition Tree Definition

1. Given a node $\Gamma \longrightarrow \Delta$

We traverse Γ from **left** to **right** to find the first decomposable formula

Its **main connective**. $\circ$ identifies a possible **decomposition rule** $(\circ \longrightarrow)$ Then we **check** if this decomposition rule $(\circ \longrightarrow)$ applies. If it does we put its conclusions (conclusion) as leaves (leaf)

2. We **traverse** Δ from **right** to **left** to find the first decomposable formula

Its **main connective** $\circ$ identifies a possible decomposition rule $(\longrightarrow \circ)$

Then we check if this decomposition rule applies. If it does we put its conclusions (conclusion) as leaves (leaf)

3. If **1** and **2** applies we **choose** one of the rules

Step 4. We repeat **Step 2.** and **Step 3.** until we obtain only leaves

Decomposition Tree Definition

Observe that a decomposable $\Gamma \longrightarrow \Delta$ is always in the domain in **one** of the decomposition rules $(\circ \longrightarrow)$, $(\longrightarrow \circ)$, or in the domain of **both**. Hence the tree $\mathbf{T}_A$ may not be unique and all possible choices of 3. give all possible decomposition trees

System **GL** Exercises

Exercise

Prove, by constructing a proper **decomposition tree** that

$$\vdash_{\mathbf{GL}} ((\neg a \Rightarrow b) \Rightarrow (\neg b \Rightarrow a))$$

Solution

By definition,we have that

$$\vdash_{\mathbf{GL}} ((\neg a \Rightarrow b) \Rightarrow (\neg b \Rightarrow a)) \text{ if and only if}$$

$$\vdash_{\mathbf{GL}} \longrightarrow ((\neg a \Rightarrow b) \Rightarrow (\neg b \Rightarrow a))$$

We construct a decomposition tree $\mathbf{T}_{\rightarrow A}$ as follows

System **GL** Exercises

$\mathbf{T}_{\rightarrow A}$

$$\longrightarrow ((\neg a \Rightarrow b) \Rightarrow (\neg b \Rightarrow a))$$

$$\mid (\rightarrow\Rightarrow)$$

$$(\neg a \Rightarrow b) \longrightarrow (\neg b \Rightarrow a)$$

$$\mid (\rightarrow\Rightarrow)$$

$$\neg b, (\neg a \Rightarrow b) \longrightarrow a$$

$$\mid (\rightarrow \neg)$$

$$(\neg a \Rightarrow b) \longrightarrow b, a$$

$$\bigwedge (\Rightarrow\longrightarrow)$$

$$\longrightarrow \ \neg a, b, a \qquad\qquad b \longrightarrow b, a$$

$$\mid (\rightarrow \neg) \qquad\qquad \textit{axiom}$$

$$a \ \longrightarrow \ b, a$$

axiom

All leaves of the tree are axioms, hence we have found the **proof** of A in **GL**

System **GL** Exercises

Exercise

Prove, by constructing proper **decomposition trees** that

$$\nvdash_{\mathbf{GL}} ((a \Rightarrow b) \Rightarrow (\neg b \Rightarrow a))$$

Solution

Observe that for some formulas A, their decomposition tree $\mathbf{T}_{\rightarrow A}$ in **GL** may **not be unique**

Hence we have to construct all possible decomposition trees to see that none of them is a proof, i.e. to see that each of them has a non axiom leaf.

We construct the decomposition trees for $\longrightarrow A$ as follows

System **GL** Exercises

$\mathbf{T}_{\mathbf{1}\rightarrow A}$

$$\longrightarrow ((a \Rightarrow b) \Rightarrow (\neg b \Rightarrow a))$$

$\mid (\rightarrow\Rightarrow)$ (*one choice*)

$$(a \Rightarrow b) \longrightarrow (\neg b \Rightarrow a)$$

$\mid (\rightarrow\Rightarrow)$ (*first of two choices*)

$$\neg b, (a \Rightarrow b) \longrightarrow a$$

$\mid (\neg \rightarrow)$ (*one choice*)

$$(a \Rightarrow b) \longrightarrow b, a$$

$\bigwedge (\Rightarrow\longrightarrow)$ (*one choice*)

$\longrightarrow a, b, a$ $\qquad\qquad$ $b \longrightarrow b, a$

non – axiom $\qquad\qquad$ *axiom*

The tree contains a **non- axiom** leaf, hence it is **not a proof**

We have one more tree to construct

System **GL** Exercises

$\mathbf{T}_{\mathbf{2}\rightarrow A}$

$$\longrightarrow ((a \Rightarrow b) \Rightarrow (\neg b \Rightarrow a))$$

$\mid (\rightarrow\Rightarrow)$ (*one choice*)

$$(a \Rightarrow b) \longrightarrow (\neg b \Rightarrow a)$$

$\bigwedge(\Rightarrow\longrightarrow)$ (*second choice*)

$\longrightarrow (\neg b \Rightarrow a), a$

$\mid (\longrightarrow\Rightarrow)$ (*one choice*)

$\neg b \longrightarrow a, a$

$\mid (\neg \rightarrow)$ (*one choice*)

$\longrightarrow a, a, b$

non – axiom

$b \longrightarrow (\neg b \Rightarrow a)$

$\mid (\rightarrow\Rightarrow)$ (*one choice*)

$b, \neg b \longrightarrow a$

$\mid (\neg \rightarrow)$ (*one choice*)

$b \longrightarrow a, b$

axiom

All possible trees end with a non-axiom leaf. It proves that

$\nvdash_{\mathbf{GL}} ((a \Rightarrow b) \Rightarrow (\neg b \Rightarrow a))$

System **GL** Exercises

Does the tree below constitute a proof in **GL** ? Justify your answer

$\mathbf{T}_{\rightarrow A}$

$$\longrightarrow \neg\neg((\neg a \Rightarrow b) \Rightarrow (\neg b \Rightarrow a))$$

$$\mid (\rightarrow \neg)$$

$$\neg((\neg a \Rightarrow b) \Rightarrow (\neg b \Rightarrow a)) \longrightarrow$$

$$\mid (\neg \rightarrow)$$

$$\longrightarrow ((\neg a \Rightarrow b) \Rightarrow (\neg b \Rightarrow a))$$

$$\mid (\rightarrow\Rightarrow)$$

$$(\neg a \Rightarrow b) \longrightarrow (\neg b \Rightarrow a)$$

$$\mid (\rightarrow\Rightarrow)$$

$$(\neg a \Rightarrow b), \neg b \longrightarrow a$$

$$\mid (\neg \rightarrow)$$

$$(\neg a \Rightarrow b) \longrightarrow b, a$$

$$\bigwedge (\Rightarrow\longrightarrow)$$

$\longrightarrow \neg a, b, a$	$b \longrightarrow b, a$
$\mid (\rightarrow \neg)$	*axiom*
$a \longrightarrow b, a$	
axiom	

Solution

The tree $\mathbf{T}_{\rightarrow A}$ is **not a proof** in **GL** because a rule corresponding to the decomposition step below **does not exists** in **GL**

$$(\neg a \Rightarrow b), \neg b \longrightarrow a$$

$$\mid (\neg \rightarrow)$$

$$(\neg a \Rightarrow b) \longrightarrow b, a$$

It is a proof is some system **GL1** that has all the rules of **GL** except its rule $(\neg \rightarrow)$

$$(\neg \rightarrow) \quad \frac{\Gamma', \Gamma \ \longrightarrow \ \Delta, \ A, \ \Delta'}{\Gamma', \ \neg A, \ \Gamma \ \longrightarrow \ \Delta, \Delta'}$$

This rule has to be replaced in by the rule:

$$(\neg \rightarrow)_1 \ \frac{\Gamma, \Gamma' \ \longrightarrow \ \Delta, A, \Delta'}{\Gamma, \neg A, \Gamma' \ \longrightarrow \ \Delta, \Delta'}$$

Exercises

Exercise 1

Write all proofs in **GL** of

$$(\neg(a \cap b) \Rightarrow (\neg a \cup \neg b))$$

Exercise 2

Find a formula which has a unique decomposition tree

Exercise 3

Describe for which kind of formulas the decomposition tree is unique

System **GL** Exercises

Exercise

We know that the system **GL** is **strongly sound**

Prove, by constructing a **counter-model** determined by a proper decomposition tree that

$$\not\models ((b \Rightarrow a) \Rightarrow (\neg b \Rightarrow a))$$

We construct the decomposition tree for the formula

$A : ((b \Rightarrow a) \Rightarrow (\neg b \Rightarrow a))$ as follows

System **GL** Exercises

$\mathbf{T}_{\rightarrow A}$

$$\longrightarrow ((b \Rightarrow a) \Rightarrow (\neg b \Rightarrow a))$$

$$\mid (\rightarrow\Rightarrow)$$

$$(b \Rightarrow a) \;\longrightarrow\; (\neg b \Rightarrow a)$$

$$\mid (\rightarrow\Rightarrow)$$

$$\neg b, (b \Rightarrow a) \;\longrightarrow\; a$$

$$\mid (\neg \rightarrow)$$

$$(b \Rightarrow a) \;\longrightarrow\; b, a$$

$$\bigwedge (\Rightarrow\longrightarrow)$$

$$\longrightarrow\; b, b, a \qquad\qquad\qquad a \longrightarrow b, a$$

non − axiom   *axiom*

The **counter model** determined by the tree $\mathbf{T}_{\rightarrow A}$ is any truth assignment v that evaluates the **non axiom leaf** $\longrightarrow b, b, a$ to F

System **GL** Exercises

Let $v : VAR \longrightarrow \{T, F\}$ be a truth assignment

By definition of semantic for sequents we have that

$v^*(\longrightarrow b, b, a) = (T \Rightarrow v(b) \cup v(b) \cup v(a))$

Hence $v^*(\longrightarrow b, b, a) = F$ if and only if

$(T \Rightarrow v(b) \cup v(b) \cup v(a)) = F$ if and only if
$v(b) = v(a) = F$

The **counter model** determined by the $\mathbf{T}_{\rightarrow A}$ is any
$v : VAR \longrightarrow \{T, F\}$ such that $v(b) = v(a) = F$

Exercise

Prove the **Completeness theorem** for GL

Assume that the Strong Soundness has been already proved and the Decompositions Trees are already defined

Reminder

Formula Completeness for GL: For any $A \in \mathcal{F}$,

$$\models A \quad \text{if and only if} \quad \vdash_{GL} \longrightarrow A$$

Soundness for GL: For any $A \in \mathcal{F}$,

$$\text{If } \vdash_{GL} \longrightarrow A, \quad \text{then } \models A$$

Completeness part for GL: For any $A \in \mathcal{F}$,

$$\text{If } \models A, \textit{ then } \vdash_{GL} \longrightarrow A$$

Proof of Completeness of **GL**

We prove the logically equivalent form of the **Completeness part:** For any $A \in \mathcal{F}$,

$$\text{If } \nvdash_{GL} \longrightarrow A \text{ then } \not\models A$$

Proof

Assume $\nvdash_{GL} \longrightarrow A$, i.e. $\longrightarrow A$ does not have a proof in **GL**

Let $\mathcal{T}_A$ be a **set of all decomposition trees** of $\longrightarrow A$

As $\nvdash_{GL} \longrightarrow A$, each $T \in \mathcal{T}_A$ has a **non-axiom leaf**

We choose an arbitrary $T_A \in \mathcal{T}_A$

Proof of Completeness of **GL**

Let $\Gamma' \longrightarrow \Delta', \Gamma', \Delta' \in VAR^*$ be the **non-axiom leaf** of the tree T_A

The non-axiom leaf $\Gamma' \longrightarrow \Delta'$ **determines** a truth assignment $v : VAR \longrightarrow \{T, F\}$ which is defined as follows:

$$v(a) = \begin{cases} T & \text{if } a \text{ appears in } \Gamma' \\ F & \text{if } a \text{ appears in } \Delta' \\ \textit{any value} & \text{if } a \text{ does not appear in } \Gamma' \longrightarrow \Delta' \end{cases}$$

By the **strong soundness** of the rules of inference of **GL** it proves that $v^*(A) = F$, i.e. that $\not\models A$

Original Gentzen systems
LK for Classical Propositional Logic and **LI** for Intuitionistic Logic

Gentzen Systems **LK, LI** for Classical and Intuitionistic Propositional Logics

The proof systems **LK** for Classical Propositional Logic and **LI** for Intuitionistic Propositional Logic as presented here are **CUT- Free** versions of original systems published by G. Gentzen in 1935.

The proof system **LI** for Intuitionistic Logic was presented as a **particular case** of his proof system **LK** for the classical logic

Both original Gentzen systems **LK, LI** were created for Predicate Logics. We present here only their Propositional version. The Predicate version to follow

Classical Gentzen System **LK**

Language of **LK**

$$\mathcal{L} = \mathcal{L}_{\{\neg, \cap, \cup, \Rightarrow\}} \quad \text{and} \quad \mathcal{E} = SQ$$

for

$$SQ = \{\Gamma \longrightarrow \Delta : \quad \Gamma, \Delta \in \mathcal{F}^*\}$$

Axioms of LK any sequent of the form

$$\Gamma_1, \ A, \Gamma_2 \quad \longrightarrow \quad \Gamma_3, \ A, \ \Gamma_4$$

Classical Gentzen System **LK**

Rules of inference of **LK** are as follows

Structural Rules

Weakening

$$(weak \rightarrow)\quad \frac{\Gamma \longrightarrow \Delta}{A, \Gamma \longrightarrow \Delta}$$

$$(\rightarrow weak)\quad \frac{\Gamma \longrightarrow \Delta}{\Gamma \longrightarrow \Delta, A}$$

Contraction

$$(contr \rightarrow)\quad \frac{A, A, \Gamma \longrightarrow \Delta}{A, \Gamma \longrightarrow \Delta}$$

$$(\rightarrow contr)\quad \frac{\Gamma \longrightarrow \Delta, A, A}{\Gamma \longrightarrow \Delta, A}$$

Classical Gentzen System **LK**

Structural Rules

Exchange

$$(exch \rightarrow) \quad \frac{\Gamma_1,\ A, B,\ \Gamma_2 \ \longrightarrow\ \Delta}{\Gamma_1,\ B, A,\ \Gamma_2 \ \longrightarrow\ \Delta}$$

$$(\rightarrow exch) \quad \frac{\Delta \longrightarrow \Gamma_1,\ A, B,\ \Gamma_2}{\Delta \longrightarrow \Gamma_1,\ B, A,\ \Gamma_2}$$

Classical Gentzen System **LK**

Logical Rules

Conjunction rules

$$(\cap \rightarrow) \quad \frac{A, B, \Gamma \longrightarrow \Delta}{(A \cap B), \Gamma \longrightarrow \Delta}$$

$$(\rightarrow \cap) \quad \frac{\Gamma \longrightarrow \Delta, A \quad ; \quad \Gamma \longrightarrow \Delta, B, \Delta}{\Gamma \longrightarrow \Delta, (A \cap B)}$$

Disjunction rules

$$(\rightarrow \cup) \quad \frac{\Gamma \longrightarrow \Delta, A, B}{\Gamma \longrightarrow \Delta, (A \cup B)}$$

$$(\cup \rightarrow) \quad \frac{A, \Gamma \longrightarrow \Delta \quad ; \quad B, \Gamma \longrightarrow \Delta}{(A \cup B), \Gamma \longrightarrow \Delta}$$

Classical Gentzen System **LK**

Implication rules

$$(\longrightarrow\Rightarrow) \qquad \frac{A,\ \Gamma \ \longrightarrow\ \Delta,\ B}{\Gamma\ \longrightarrow\ \Delta,\ (A \Rightarrow B)}$$

$$(\Rightarrow\longrightarrow) \qquad \frac{\Gamma\ \longrightarrow\ \Delta,\ A \quad ; \quad B,\ \Gamma\ \longrightarrow\ \Delta}{(A \Rightarrow B),\ \Gamma\ \longrightarrow\ \Delta}$$

Classical Gentzen System **LK**

Negation rules

$$(\neg \longrightarrow) \quad \frac{\Gamma \longrightarrow \Delta, A}{\neg A, \Gamma \longrightarrow \Delta}$$

$$(\longrightarrow \neg) \quad \frac{A, \Gamma \longrightarrow \Delta}{\Gamma \longrightarrow \Delta, \neg A}$$

We define formally

$$\textbf{LK} = (\mathcal{L}, SQ, \ AX, \quad \text{Structural rules}, \ \ \text{Logical rules})$$

Intuitionistic I Gentzen System **LI**

Language of LI

Any expression

$$\Gamma \longrightarrow \Delta$$

where $\Gamma, \Delta \in \mathcal{F}^*$ and

Δ consists of **at most one formula**

is called a **LI** sequent

We denote the set of all **LI** sequents by *ISQ*, i.e.

$$ISQ = \{\Gamma \longrightarrow \Delta : \quad \Delta \quad \text{consists of} \quad \textbf{at most one formula}\}$$

Axioms of **LI**

Axioms of LI consist of any sequent from the set *ISQ* which contains a formula that appears on both sides of the sequent arrow $\longrightarrow$, i.e any sequent of the form

$$\Gamma,\ A,\ \Delta\ \longrightarrow\ A$$

for $\Gamma, \Delta \in \mathcal{F}^*$

Rules of Inference of **LI**

The set inference rules of **LI** is divided into two groups : the **structural rules** and the **logical rules**

There are three **Structural Rules** of **LI**: Weakening, Contraction and Exchange

Weakening structural rule

$$(weak \rightarrow) \quad \frac{\Gamma \longrightarrow \Delta}{A, \Gamma \longrightarrow \Delta}$$

$$(\rightarrow weak) \quad \frac{\Gamma \longrightarrow}{\Gamma \longrightarrow A}$$

A is called the weakening formula

Remember that Δ contains at most one formula

Rules of Inference of **LI**

Contraction structural rule

$$(contr \rightarrow) \quad \frac{A, A, \Gamma \longrightarrow \Delta}{A, \Gamma \longrightarrow \Delta}$$

The case below is **not VALID** for **LI**; we list it as it will be used in the classical case

$$(\rightarrow contr) \quad \frac{\Gamma \longrightarrow \Delta, A, A}{\Gamma \longrightarrow \Delta, A}$$

A is called the contraction formula

Remember that Δ contains at most one formula

Rules of Inference of **LI**

Exchange structural rule

$$(exch \rightarrow) \quad \frac{\Gamma_1,\; A, B,\; \Gamma_2 \;\longrightarrow\; \Delta}{\Gamma_1,\; B, A,\; \Gamma_2 \;\longrightarrow\; \Delta}$$

The case below is **not VALID** for **LI**; we list it as it will be used in the classical case

$$(\rightarrow exch) \quad \frac{\Delta \longrightarrow \Gamma_1,\; A, B,\; \Gamma_2}{\Delta \longrightarrow \Gamma_1,\; B, A,\; \Gamma_2}.$$

Remember that Δ contains at most one formula

Rules of Inference of **LI**

Logical Rules

Conjunction rules

$$(\cap \rightarrow) \quad \frac{A, B, \Gamma \longrightarrow \Delta}{(A \cap B), \Gamma \longrightarrow \Delta},$$

$$(\rightarrow \cap) \quad \frac{\Gamma \longrightarrow A \; ; \; \Gamma \longrightarrow B}{\Gamma \longrightarrow (A \cap B)}$$

Remember that Δ contains at most one formula

Rules of Inference of **LI**

Disjunction rules

$$(\rightarrow \cup)_1 \quad \frac{\Gamma \longrightarrow A}{\Gamma \longrightarrow (A \cup B)}$$

$$(\rightarrow \cup)_2 \quad \frac{\Gamma \longrightarrow B}{\Gamma \longrightarrow (A \cup B)}$$

$$(\cup \rightarrow) \quad \frac{A, \Gamma \longrightarrow \Delta \quad ; \quad B, \Gamma \longrightarrow \Delta}{(A \cup B), \Gamma \longrightarrow \Delta}$$

Remember that Δ contains at most one formula