cse371/mat371
LOGIC

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LECTURE 11

Chapter 11
Formal Theories and Gödel Theorems

Chapter 10
Formal Theories and Gödel Theorems

Introduction

Formal theories play crucial role in mathematics and were historically defined for classical predicate (first order logic) and consequently for other first and higher order logics, classical and non-classical

The idea of formalism in mathematics, which resulted in the concept of **formal theories**, or formalized theories, as they are also called

The concept of Formal theories was developed in connection with the **Hilbert Program**

Introduction

One of the main objects of the Hilbert Program was to construct a **formal theory** that would cover the whole mathematics and to prove its consistency by employing the **simplest** of logical means.

This part of the program was called the Consistency Program

Consistent Theory

A **formal theory** is said to be consistent if no formal proof can be carried in that theory for a formula A and at the same time for its negation $\neg A$.

Introduction

In 1930 , while still in his twenties **Kurt Gödel** made a historic announcement:

Hilbert Consistency Program could not be carried out

Gödel justified his claim by proving his **Inconsistency Theorem**, called also **Second Incompleteness Theorem**

Roughly speaking the **Inconsistency Theorem** states that a proof of the consistency of every **formal theory** that contains arithmetic of natural numbers can be carried out **only** in mathematical theory which is more comprehensive than the one whose **consistency** is to be proved

Introduction

In particular, a proof of the **consistency** of formal (elementary, first order) arithmetic of natural numbers can be **carried out only** in mathematical theory which contains the whole arithmetic **and also** other theorems that do not belong to arithmetic

It applies to a **formal theory** that would cover the whole mathematics because it would obviously contain the arithmetic on natural numbers

Hence the Hilbert Consistency Program **fails**

Introduction

Gödel result concerning the proofs of the consistency of formal mathematical theories has had a decisive impact on research in properties of **formal theories**

Instead of looking for direct proofs of **inconsistency** of mathematical theories mathematicians concentrated largely to relative proofs

The relative proofs demonstrate that a **theory** under consideration is **consistent** if a certain other theory, for example a formal theory of natural numbers is **consistent**

Introduction

All those **relative proofs** are rooted in a deep conviction that even though it cannot be proved that the theory of natural numbers is free of inconsistencies, it is **consistent**

This conviction is confirmed by **centuries** of development of mathematics and experiences of mathematicians

Complete Theories

A **formal theory** is called complete if for every sentence (formula without free variables) of the language of that theory **there is** a formal proof of it **or** of its negation.

A formal theory which does not have this property is called incomplete

Hence a **formal theory** is incomplete if **there is** a sentence A of the language of that theory, such that **neither** A **nor** $\neg A$ are **provable in it**

Such sentences are called **undecidable** in the **theory** in question or independent of the **theory**

Introduction

It might seem that one should be able to formalize a theory such as the formal theory of natural numbers in a way to make it **complete**

But it is not the case in view of **Gödel Incompleteness Theorem** that states:

Every **consistent** *formal theory which contains the arithmetic of natural numbers is* **incomplete**

Gödel Inconsistency Theorem follows form it

This is why the Incompleteness and Inconsistency Theorems are now called **Gödel First Incompleteness Theorem** and **Gödel Second Incompleteness Theorem**, respectively.

Peano Arithmetic PA

Formal Theory of Natural Numbers

Peano Arithmetic PA

Next to geometry, the **theory of natural numbers** in the most intuitive and intuitively known of all branches of mathematics

This is why the first attempts to **formalize mathematics** begin with with arithmetic of natural numbers.

The first attempt of axiomatic formalization was given by Dedekind in **1879** and by Peano in **1889**

The Peano formalization became known as **Peano Postulates** (axioms) and can be written as follows.

Peano Arithmetic PA

Peano Postulates

p1 0 is a natural number

p2 If n is a natural number, there is another number which we denote by n'

We call n' a **successor** of n

The intuitive meaning of n' is $n+1$

p3 $0 \neq n'$, for any natural number n

p4 If $n' = m'$, then $n = m$, for any natural numbers n, m

Peano Arithmetic PA

p5 If W is is a property that may or may not hold for natural numbers, and

if **(i)** 0 has the property W and

(ii) whenever a natural number n has the property W, then n' has the property W,

then all natural numbers have the property W

The postulate **p5** is called *Principle of Induction*

These axioms together with certain amount of set theory are sufficient to develop not only theory of natural numbers, but also theory of rational and even real numbers.

But they can't act as a fully formal theory as they include intuitive notions like "property" and "has a property".

FORMAL Peano Arithmetic PA

Language of PA

$$\mathcal{L}_{PA} = \mathcal{L}(\mathbf{P} = \{P\},\ \mathbf{F} = \{f, g, h\},\ \mathbf{C} = \{c\}),$$

where # P = 2, i.e. P is a two argument predicate.

The **intended interpretation** of P is **equality** so we use the equality symbol $=$ instead of P

We write $x = y$ instead $= (x, y)$

We write $x \neq y$ for $\neg(x = y)$

f is a one argument functional symbol; $f(x)$ represent the successor of a given x and we denote it by x'

FORMAL Peano Arithmetic PA

g, h is are two argument functional symbols

The **intended interpretation** of f is addition and the **intended interpretation** of g is multiplication

We write $x + y$ for $f(x, y)$ and $x \cdot y$ for $g(x, y)$

c is a constant symbol representing zero and we use a symbol 0 to denote c

We write the language of PA as

$$\mathcal{L}_{PA} = \mathcal{L}_{\{\neg, \Rightarrow, \cup, \cap\}}(\{=\},\ \{\ ',\ +,\ \cdot\},\ \{0\})$$

FORMAL Peano Arithmetic PA

Specific Axioms

P1 $(x = y \Rightarrow (x = z \Rightarrow y = z))$,

P2 $(x = y \Rightarrow x' = y')$,

P3 $0 \neq x'$,

P4 $(x' = y' \Rightarrow x = y)$,

P5 $x + 0 = x$,

P6 $x + y' = (x + y)'$

P7 $x \cdot 0 = 0$,

P8 $x \cdot y' = (x \cdot y) + x$,

P9 $(A(0) \Rightarrow (\forall x(A(x) \Rightarrow A(x') \Rightarrow \forall x A(x))))$,

for all formulas $A(x)$ of $\mathcal{L}_{PA}$ and all $x, y, z \in VAR$

FORMAL Peano Arithmetic PA

The axiom P9 is called Principle of Mathematical Induction

It does not fully corresponds to Peano Postulate **p5** which refers intuitively to all possible properties on natural numbers (uncountably many)

TheP7 axiom applies only to properties defined by infinitely countably formulas of $A(x)$ of $\mathcal{L}_{PA}$

Axioms P3, P4 correspond to Peano Postulates **p3, p4**

The Postulates **p1, p2** are taken care by presence of0 and successor function

Axioms P1, P2 deal with some needed properties of equality that were probably assumed as intuitively obvious by Peano and Dedekind

FORMAL Peano Arithmetic PA

Axioms P5 - P8 are the recursion equations for addition and multiplication

They are not stated in the Peano Postulates as Dedekind and Peano allowed the use of **intuitive set theory** within which the existence of addition and multiplication and their properties P5-P8 can be proved (Mendelson, 1973)

Gödel THEOREMS

First Incompleteness Theorem

Let T be a formal theory containing arithmetic

Then there is a sentence A in the language of T which **asserts its own unprovability** and is such that:

(i) If T is consistent, then $\nvdash_T A$

—bf (ii) If T is ω- consistent, then $\nvdash_T \neg A$

Gödel THEOREMS

Second Incompleteness Theorem]

Let T be a consistent formal theory containing arithmetic

Then

$$\nvdash_T \ Con_T$$

where Con_T is the sentence in the language of T asserting the consistency of T