CSE/MAT371 QUIZ 2 SOLUTIONS Fall 2017

QUESTION 1

Consider a **strongly sound** system **RS'** obtained from **RS** by changing the sequence Γ' into Γ and Δ into Δ' in all of the rules of inference of **RS**.

1. Construct a decomposition tree (of your choice) of a formula A: $((a \Rightarrow b) \Rightarrow (\neg b \Rightarrow a))$

Solution Here it decomposition tree **T** with the possible decomposition choices marked and chosen. Your Tree might be different!

$$\mathbf{T}_A$$

$$((a \Rightarrow b) \Rightarrow (\neg b \Rightarrow a))$$

$$one\ choice$$

$$\mid (\Rightarrow)$$

$$\neg(a \Rightarrow b),\ (\neg b \Rightarrow a)$$

$$two\ choices:\ \textbf{first formula choice}$$

$$\bigwedge(\neg \Rightarrow$$

$a,\ (\neg b \Rightarrow a)$	$\neg b,\ (\neg b \Rightarrow a$
one choice	*one choice*
$\mid (\Rightarrow)$	$\mid (\Rightarrow)$
$a,\ \neg\neg b, a$	$\neg b,\ \neg\neg b, a$
one choice	*one choice*
$\mid (\neg\neg)$	$\mid (\neg\neg$
$a,\ b, a$	$\neg b,\ b, a$
non axiom	*axiom*

The tree contains a **non- axiom** leaf, hence it is **not a proof**.

2. Define in your own words, for any A, the decomposition tree $\mathbf{T}_A$ in **RS'**.

Solution The definition of the decomposition tree $\mathbf{T}_A$ is in its essence similar to the one for **RS**, except for the changes which reflect the **difference** in the corresponding rules of decomposition. The tree $\mathbf{T}_A$ is not, as in the case of **RS** uniquely determined by the formula A.

We follow now the following steps

Step 1 Decompose A using a rule defined by its main connective.

Step 2 Traverse resulting sequence Γ on the new node of the tree from **right** to **left** or **left** to **right** and **find** the first decomposable formula.

Step 3 Repeat **Step 1** and **Step 2** until no more decomposable formulas

End of Tree Construction

3. Prove Completeness Theorem for **RS'**.

Assume $\nvdash_{RS"} A$. Then **every** decomposition tree of A has at least one non-axiom leaf. Otherwise, there would exist a tree with all axiom leaves and it would be a proof for A. Let $\mathcal{T}_A$ be a set of all decomposition trees of A. We choose an arbitrary $T_A \in \mathcal{T}_A$ with at least one non-axiom leaf L_A. We use the non-axiom leaf L_A to define a truth assignment v which falsifies A, as follows:

$$v(a) = \begin{cases} F & \text{if a appears in } L_A \\ T & \text{if } \neg a \text{ appears in } L_A \\ \text{any value} & \text{if a does not appear in } L_A \end{cases}$$

The value for a sequence that corresponds to the leaf in is F. Since, because of the strong soundness F "climbs" the tree, we found a counter-model for A. This proves that $\not\models A$. Part 2. proof is identical to the proof in **RS** case.

QUESTION 2

Let **GL** be the Gentzen style proof system for classical logic.

Prove, by constructing a proper decomposition tree that $\vdash_{\mathbf{GL}}((\neg(a \cap b) \Rightarrow b) \Rightarrow (\neg b \Rightarrow (\neg a \cup \neg b)))$.

Solution THIS IS NOT THE ONLY SOLUTION!

$$\mathbf{T}_{\to A}$$

$$\longrightarrow ((\neg(a \cap b) \Rightarrow b) \Rightarrow (\neg b \Rightarrow (\neg a \cup \neg b)))$$

$$\mid (\to\Rightarrow)$$

$$(\neg(a \cap b) \Rightarrow b) \longrightarrow (\neg b \Rightarrow (\neg a \cup \neg b))$$

$$\mid (\to\Rightarrow)$$

$$\neg b, (\neg(a \cap b) \Rightarrow b) \longrightarrow (\neg a \cup \neg b)$$

$$\mid (\to \cup)$$

$$\neg b, (\neg(a \cap b) \Rightarrow b) \longrightarrow \neg a, \neg b$$

$$\mid (\to \neg)$$

$$b, \neg b, (\neg(a \cap b) \Rightarrow b) \longrightarrow \neg a$$

$$\mid (\to \neg)$$

$$b, a, \neg b, (\neg(a \cap b) \Rightarrow b) \longrightarrow$$

$$\mid (\neg \to)$$

$$b, a, (\neg(a \cap b) \Rightarrow b) \longrightarrow b$$

$$\bigwedge(\Rightarrow\to$$

$b, a \longrightarrow \neg(a \cap b), b$	$b, a, b \longrightarrow b$
$\mid (\to \neg)$	*axiom*
$b, a, (a \cap b) \longrightarrow b$	
$\mid (\cap \to)$	
$b, a, a, b \longrightarrow b$	
axiom	

All leaves of the decomposition tree are axioms, hence the proof has been found.

QUESTION 3

We know that **GL** is **strongly sound**, use a decomposition tree $\mathbf{T}_{\rightarrow A}$ to construct a **counter model** for a formula

$$((a \Rightarrow b) \Rightarrow (\neg b \Rightarrow a))$$

Solution This is not the only correct Tree! (5pts)

$$\mathbf{T}_{\rightarrow A}$$

$$\longrightarrow ((a \Rightarrow b) \Rightarrow (\neg b \Rightarrow a))$$

$$\mid (\rightarrow\Rightarrow)$$

$$(a \Rightarrow b) \longrightarrow (\neg b \Rightarrow a)$$

$$\mid (\rightarrow\Rightarrow)$$

$$\neg b, (a \Rightarrow b) \longrightarrow a$$

$$\mid (\rightarrow\Rightarrow)$$

$$(a \Rightarrow b) \longrightarrow b, a$$

$$\bigwedge (\Rightarrow\rightarrow$$

$$\begin{array}{cc} \longrightarrow a, b, a & b \longrightarrow b, a \\ non - axiom & axiom \end{array}$$

(5pts) The **counter-model** determined by $\mathbf{T}_{\rightarrow A}$ is any truth assignment v that evaluates the non axiom leaf $\longrightarrow b, b, a$ to F.

By the **strong soundness**, the value F "climbs the tree" and we get that also $v * (A) = F$.

We evaluate $v^*(\longrightarrow b, b, a) = (T \Rightarrow v(b) \cup v(b) \cup v(a)) = F$ if and only if $v(b) = v(a) = F$.

The **counter model** determined by the tree $\mathbf{T}_{\rightarrow A}$ is any $v : VAR \longrightarrow \{T, F\}$ such that $v(b) = v(a) = F$

Extra Credit

We know that a classical tautology $(\neg(a \cap b) \Rightarrow (\neg a \cup \neg b))$ is NOT Intuitionistic tautology and we know by **Tarski Theorem** that $\neg\neg(\neg(a \cap b) \Rightarrow (\neg a \cup \neg b))$ is intuitionistically PROVABLE

FIND the proof of the formula

$$\neg\neg(\neg(a \cap b) \Rightarrow (\neg a \cup \neg b))$$

in the Gentzen system **LI** for Intuitionistic Logic.

Solution

$\mathbf{T}_{\rightarrow A}$

$$\longrightarrow \neg\neg(\neg(a \cap b) \Rightarrow (\neg a \cup \neg b))$$

$$\mid (\rightarrow \neg)$$

$$\neg(\neg(a \cap b) \Rightarrow (\neg a \cup \neg b)) \longrightarrow$$

$$\mid (contr \rightarrow)$$

$$\neg(\neg(a \cap b) \Rightarrow (\neg a \cup \neg b)), \neg(\neg(a \cap b) \Rightarrow (\neg a \cup \neg b)) \longrightarrow$$

$$\mid (\neg \rightarrow)$$

$$\neg(\neg(a \cap b) \Rightarrow (\neg a \cup \neg b)) \longrightarrow (\neg(a \cap b) \Rightarrow (\neg a \cup \neg b))$$

$$\mid (\rightarrow \Rightarrow)$$

$$\neg(a \cap b), \neg(\neg(a \cap b) \Rightarrow (\neg a \cup \neg b)) \longrightarrow (\neg a \cup \neg b)$$

$$\mid (\rightarrow \cup)_1$$

$$\neg(a \cap b), \neg(\neg(a \cap b) \Rightarrow (\neg a \cup \neg b)) \longrightarrow \neg a$$

$$\mid (\rightarrow \neg)$$

$$a, \neg(a \cap b), \neg(\neg(a \cap b) \Rightarrow (\neg a \cup \neg b)) \longrightarrow$$

$$\mid (exch \rightarrow)$$

$$\neg(a \cap b), a, \neg(\neg(a \cap b) \Rightarrow (\neg a \cup \neg b)) \longrightarrow$$

$$\mid (\neg \rightarrow)$$

$$a, \neg(\neg(a \cap b) \Rightarrow (\neg a \cup \neg b)) \longrightarrow (a \cap b)$$

$$\bigwedge (\rightarrow \cap$$

$$a, \neg(\neg(a \cap b) \Rightarrow (\neg a \cup \neg b)) \longrightarrow a)$$

axiom

$$a, \neg(\neg(a \cap b) \Rightarrow (\neg a \cup \neg b)) \longrightarrow b$$

$$\mid (\rightarrow weak)$$

$$a, \neg(\neg(a \cap b) \Rightarrow (\neg a \cup \neg b)) \longrightarrow$$

$$\mid (exch \rightarrow)$$

$$\neg(\neg(a \cap b) \Rightarrow (\neg a \cup \neg b)), a \longrightarrow$$

$$\mid (\neg \rightarrow)$$

$$a \longrightarrow (\neg(a \cap b) \Rightarrow (\neg a \cup \neg b))$$

$$\mid (\rightarrow \Rightarrow)$$

$$\neg(a \cap b), a \longrightarrow (\neg a \cup \neg b)$$

$$\mid (\rightarrow \cup)_2$$

$$\neg(a \cap b), a \longrightarrow \neg b$$

$$\mid (\rightarrow \neg)$$

$$b, \neg(a \cap b), a \longrightarrow$$

$$\mid (exch \rightarrow)$$

$$\neg(a \cap b), b, a \longrightarrow$$

$$\mid (\neg \rightarrow)$$

$$b, a \longrightarrow (a \cap b)$$

$$\bigwedge (\rightarrow \cap$$

$$b, a \longrightarrow a \qquad\qquad b, a \longrightarrow b$$

axiom *axiom*

All leaves are axioms, the tree is a proof of A in **LI**.

1 GL Proof System

Axioms of GL

$$\Gamma'_1, a, \Gamma'_2 \longrightarrow \Delta'_1, a, \Delta'_2, \tag{1}$$

for any $a \in VAR$ and any sequences $\Gamma'_1, \Gamma'_2, \Delta'_1, \Delta'_2 \in VAR^*$.
Inference rules of GL
The inference rules of **GL** are defined as follows.
Conjunction rules

$$(\cap \rightarrow)\ \frac{\Gamma', A, B, \Gamma \longrightarrow \Delta'}{\Gamma', (A \cap B), \Gamma \longrightarrow \Delta'}, \qquad (\rightarrow \cap)\ \frac{\Gamma \longrightarrow \Delta, A, \Delta' \ ; \ \Gamma \longrightarrow \Delta, B, \Delta'}{\Gamma \longrightarrow \Delta, (A \cap B), \Delta'},$$

Disjunction rules

$$(\rightarrow \cup)\ \frac{\Gamma \longrightarrow \Delta, A, B, \Delta'}{\Gamma \longrightarrow \Delta, (A \cup B), \Delta'}, \qquad (\cup \rightarrow)\ \frac{\Gamma', A, \Gamma \longrightarrow \Delta' \ ; \ \Gamma', B, \Gamma \longrightarrow \Delta'}{\Gamma', (A \cup B), \Gamma \longrightarrow \Delta'},$$

Implication rules

$$(\rightarrow\Rightarrow)\ \frac{\Gamma', A, \Gamma \longrightarrow \Delta, B, \Delta'}{\Gamma', \Gamma \longrightarrow \Delta, (A \Rightarrow B), \Delta'}, \qquad (\Rightarrow\rightarrow)\ \frac{\Gamma', \Gamma \longrightarrow \Delta, A, \Delta' \ ; \ \Gamma', B, \Gamma \longrightarrow \Delta, \Delta'}{\Gamma', (A \Rightarrow B), \Gamma \longrightarrow \Delta, \Delta'},$$

Negation rules

$$(\neg \rightarrow)\ \frac{\Gamma', \Gamma \longrightarrow \Delta, A, \Delta'}{\Gamma', \neg A, \Gamma \longrightarrow \Delta, \Delta'}, \qquad (\rightarrow \neg)\ \frac{\Gamma', A, \Gamma \longrightarrow \Delta, \Delta'}{\Gamma', \Gamma \longrightarrow \Delta, \neg A, \Delta'}.$$

2 LI Proof System

Axioms of LI
As the axioms of **LI** we adopt any sequent of the form

$$\Gamma_1, A, \Gamma_2 \longrightarrow A$$

for any formula $A \in \mathcal{F}$ and any sequences $\Gamma_1, \Gamma_2 \in \mathcal{F}^*$.
Inference rules of LI
The set inference rules is divided into two groups: the structural rules and the logical rules. They are defined as follows.
Structural Rules of LI
Weakening

$$(\rightarrow weak)\ \frac{\Gamma \longrightarrow}{\Gamma \longrightarrow A}\ .$$

A is called the weakening formula.
Contraction

$$(contr \rightarrow)\ \frac{A, A, \Gamma \longrightarrow \Delta}{A, \Gamma \longrightarrow \Delta},$$

A is called the contraction formula , Δ contains at most one formula.
Exchange

$$(exchange \rightarrow)\ \frac{\Gamma_1, A, B, \Gamma_2 \longrightarrow \Delta}{\Gamma_1, B, A, \Gamma_2 \longrightarrow \Delta},$$

Δ contains at most one formula.

Logical Rules of LI
Conjunction rules

$$(\cap \rightarrow)\ \frac{A, B, \Gamma \longrightarrow \Delta}{(A \cap B), \Gamma \longrightarrow \Delta}, \qquad (\rightarrow \cap)\ \frac{\Gamma \longrightarrow A\ ;\ \Gamma \longrightarrow B}{\Gamma \longrightarrow (A \cap B)},$$

Δ contains at most one formula.
Disjunction rules

$$(\rightarrow \cup)_1\ \frac{\Gamma \longrightarrow A}{\Gamma \longrightarrow (A \cup B)}, \qquad (\rightarrow \cup)_2\ \frac{\Gamma \longrightarrow B}{\Gamma \longrightarrow (A \cup B)},$$

$$(\cup \rightarrow)\ \frac{A, \Gamma \longrightarrow \Delta\ ;\ B, \Gamma \longrightarrow \Delta}{(A \cup B), \Gamma \longrightarrow \Delta},$$

Δ contains at most one formula.
Implication rules

$$(\rightarrow\Rightarrow)\ \frac{A, \Gamma \longrightarrow B}{\Gamma \longrightarrow (A \Rightarrow B)}, \qquad (\Rightarrow\rightarrow)\ \frac{\Gamma \longrightarrow A\ ;\ B, \Gamma \longrightarrow \Delta}{(A \Rightarrow B), \Gamma \longrightarrow \Delta},$$

Δ contains at most one formula.
Negation rules

$$(\neg \rightarrow)\ \frac{\Gamma \longrightarrow A}{\neg A, \Gamma \longrightarrow}, \qquad (\rightarrow \neg)\ \frac{A, \Gamma \longrightarrow}{\Gamma \longrightarrow \neg A}.$$