CSE/MAT371 Midterm 1 SOLUTIONS Fall 2016
(20pts)

NAME **ID:** **Math/CS**

QUESTION 1 (10pts)

(a) Give example (by name) of three non-classical logics.

Solution: Intuitionistic Logic, Modal Logic S4, S5, and any of CS logics listed below.

(b) Give example (by name) of two logics developed by computer scientists.

Solution: *Dynamic logic* (Harel 1979) which was created to facilitate the statement and proof of properties of programs.

Temporal Logics which were created for the specification and verification of concurrent programs Harel, Parikh, 1979, 1983 and for a specification of hardware circuits Halpern, Manna and Maszkowski, (1983).

Fuzzy logic, Many valued logics that were created and developed to describe reasoning with incomplete information.

Non-monotonic logics were created by Mc Carthy (1985) and has been shown to be important in other areas. There are applications to logic programming, to planning and reasoning about action, and to automated diagnosis.

QUESTION 2 (20pts)

Given a mathematical statement **S** written with logical symbols: $(\exists_{x>0}\ x \leq 5 \ \Rightarrow\ \forall_{x>0}\ x \in N)$.

1. Translate **S** it into a proper logical formula that **uses** restricted domain quantifiers.

Solution: The corresponding **atomic formulas** of $\mathcal{L}$ are:

$G(x, c_1)$, $L(x, c_1)$, $N(x)$ for $x > 0$, $x \leq 5$, $x \in N$, respectively.

The statement **S** becomes **restricted quantifiers** formula $\exists_{G(x,c_1)} L(x, c_1) \Rightarrow \forall_{G(x,c_1)} N(x))$

2. Translate your restricted quantifiers formula into a correct formula **without** restricted domain quantifiers.

Solution: by the **transformation** rules we get $A \in \mathcal{F}$: $(\exists x(G(x, c_1) \cap L(x, c_1)) \Rightarrow \forall x(G(x, c_1) \Rightarrow N(x)))$.

QUESTION 3 (20pts)

Given a formula A : $\exists x \forall y\ P(x, y)$ of the predicate language $\mathcal{L}$, and two model structures $\mathbf{M_1} = (N, I_1)$, $\mathbf{M_2} = (Z, I_2)$ with the interpretations $P_{I_1} : \leq$ and $P_{I_2} : >$.

1. Show that $\mathbf{M_1} \models A$.

Solution: the formula A becomes under the interpretation I_1 in $\mathbf{M_1} = (N, I_1)$ a mathematical statement $\exists x \forall y\ x \leq y$ defined in the set N of natural numbers. Hence $A_{I_1} : \exists_{x \in N} \forall_{y \in N}\ x \leq y$ asserts that there is a smallest natural number what is is **true**. It proves that $\mathbf{M_1} \models A$.

2. Show that $\mathbf{M_2} \not\models A$

Solution: the formula A becomes under the interpretation I_2 in $\mathbf{M_2} = (Z, I_2)$ a mathematical statement $\exists x \forall y\ x > y$ defined in the set Z of integers. Obviously $A_{I_2} : \exists_{x \in Z} \forall_{x \in Z}\ x > y$ is **false** as it asserts that there is a greatest integer.

Extra Credit (10 extra points)

Prove that the inverse implication to the basic predicate tautology $(\forall x\, A(x) \Rightarrow \exists x\, A(x))$ is not a predicate tautology.

It means you have to provide an example of a concrete formula $A(x)$ and construct a counter-model $\mathbf{M} = (U, I)$ for the formula $F : (\exists x\, A(x) \Rightarrow \forall x\, A(x))$.

Solution Let $A(x)$ be an atomic formula $P(x, c)$. We take as $\mathbf{M} = (N, I)$ for N set of natural numbers and $P_I :<$, $c_I : 3$. The formula F becomes an obviously *false* mathematical statement $F_I : (\exists_{n \in N} n < 3 \Rightarrow \forall_{n \in N} n < 3)$.

QUESTION 4 (10pts)

Write the following natural language statement:

One likes to play bridge, or from the fact that the weather is good we conclude the following: one does not like to play bridge or one likes not to play bridge

as a formula of 2 different languages.

1. A formula A_1 of a language $\mathcal{L}_{\{\neg,\, \mathbf{L},\, \cup,\, \Rightarrow\}}$, where **L** A represents statement "one likes A", "A is liked".

Solution: we adopt propositional variables: a, b, where a denotes statement: *play bridge*, b denotes a statement: *the weather is good*.

Translation 1: A_1 : $(\mathbf{L}a \cup (b \Rightarrow (\neg \mathbf{I}a \cup \mathbf{L}\neg a)))$

2. A formula A_2 of a language $\mathcal{L}_{\{\neg,\, \cup,\, \Rightarrow\}}$.

Solution: we adopt propositional variables a, b, c, where a denotes statement: *one likes to play bridge* , b denotes a statement: *the weather is good*, and c denotes a statement: *one likes not to play bridge*

Translation 2: A_2 : $(a \cup (b \Rightarrow (\neg a \cup c)))$.

QUESTION 5 (20pts)

Let A be a formula $A = ((((a \cap \neg c) \Rightarrow \neg b) \cup a) \Rightarrow (c \cup b))$.

1. A language $\mathcal{L}_{CON}$ to which the formula A belongs is:

Solution: The language is $\mathcal{L}_{\{\neg, \cap, \Rightarrow\}}$.

2. Determine the degree of A and write down all its sub-formulas of the degree 2.

Solution: The degree of A is 7. There is only one sub-formula of the degree 2: $(a \cap \neg c)$.

3. Determine whether : $A \in \mathbf{T}$. Use "proof by contradiction" method and **shorthand** notation.

Solution: of the case $A \in \mathbf{T}$.
Assume $((((a \cap \neg c) \Rightarrow \neg b) \cup a) \Rightarrow (c \cup b)) = F$. This is possible if and only if $(((a \cap \neg c) \Rightarrow \neg b) \cup a) = T$ and $(c \cup b) = F$. This gives as that $c = F, b = F$. We evaluate $(((a \cap \neg F) \Rightarrow \neg F) \cup a) = T$. This is possible for $a = T$. Any truth assignment such that $a = T, b = F, c = F$ is a counter-model for A, hence $A \notin \mathbf{T}$.

4. Determine whether $A \in \mathbf{C}$.

Solution: any truth assignment such that $a = T, b = T, c = F$ is a model for A, hence $A \notin \mathbf{C}$. This is not the only model.

Extra Credit (10 extra points)

Find an infinite number of formulas that are **independent** of a set $\mathcal{G}$ below. Use shorthand notation.

$$\mathcal{G} = \{((a \Rightarrow a \cup b)),\ (a \cup b),\ \neg b,\ (c \Rightarrow b)\}$$

Solution: first we find a restricted model for $\mathcal{G}$. The formula $((a \Rightarrow a \cup b))$, hence any v is its model. $\neg b = T$ only if b=F. We evaluate $(a \cup b) = (a \cup F) = T$ only if a=T. Consequently, $(c \Rightarrow b) = (c \Rightarrow F) = T$ only if c=F. Hence, any v, such that a=T, b= T, and c= F is a model for $\mathcal{G}$. Let A be any atomic formula $d \in VAR - \{a, b, c\}$. Any v, such that a=T, b= T, and c= F, d= T is a model for $\mathcal{G} \cup \{A\}$. Any v, such that a=T, b= T, and c= F, d= F is a model for $\mathcal{G} \cup \{\neg A\}$. There is countably infinitely many atomic formulas A=d where $d \in VAR - \{a, b, c\}$.

QUESTION 6 (25pts)

Given a language $\mathcal{L} = \mathcal{L}_{\{\neg, \Rightarrow, \cup, \cap\}}$. We a define a $\mathbf{L}_4$ **semantics** as follows.

Logical values are $F, \perp_1, \perp_2, T$ and they are ordered: $F < \perp_1 < \perp_2 < T$.

The **connectives** are:

$\neg$: $\{F, \perp_1, \perp_2, T\} \longrightarrow \{F, \perp_1, \perp_2, T\}$, such that $\neg\perp_1 = \perp_1$, $\neg\perp_2 = \perp_2$, $\neg F = T$, $\neg T = F$.

$\cap$: $\{F, \perp_1, \perp_2, T\} \times \{F, \perp_1, \perp_2, T\} \longrightarrow \{F, \perp_1, \perp_2, , T\}$, such that for any $x, y \in \{F, \perp_1, \perp_2, T\}$, $x \cap y = min\{x, y\}$.

$\cup$: $\{F, \perp_1, \perp_2, T\} \times \{F, \perp_1, \perp_2, T\} \longrightarrow \{F, \perp_1, \perp_2, T\}$, such that for any $x, y \in \{F, \perp_1, \perp_2, T\}$, $x \cup y = max\{x, y\}$.

$\Rightarrow$: $\{F, \perp_1, \perp_2, T\} \times \{F, \perp_1, \perp_2, T\} \longrightarrow \{F, \perp_1, \perp_2, T\}$, such that for any $x, y \in \{F, \perp_1, \perp_2, T\}$,

$$x \Rightarrow y = \begin{cases} \neg x \cup y & \text{if } x > y \\ T & \text{otherwise} \end{cases}$$

1. Write Truth Tables for **implication** and **negation**.

Solution:

$\Rightarrow$	F	$\perp_1$	$\perp_2$	T
F	T	T	T	T
$\perp_1$	$\perp_1$	T	T	T
$\perp_2$	$\perp_2$	$\perp_2$	T	T
T	F	$\perp_1$	$\perp_2$	T

$\neg$	F	$\perp_1$	$\perp_2$	T
	T	$\perp_1$	$\perp_2$	F

2. Prove $\not\models_{\mathbf{L}_4} ((a \Rightarrow b) \Rightarrow (\neg a \cup b))$. Use **shorthand** notation.

Solution: let v be a truth assignment such that $v(a) = v(b) = \perp_1$.

We evaluate $v^*((a \Rightarrow b) \Rightarrow (\neg a \cup b)) = ((\perp_1 \Rightarrow \perp_1) \Rightarrow (\neg\perp_1 \cup \perp_1)) = (T \Rightarrow (\perp_1 \cup \perp_1)) = (T \Rightarrow \perp_1) = \perp_1$.

This proves that v is a **counter-model** for our formula and that $\not\models_{\mathbf{L}_4} ((a \Rightarrow b) \Rightarrow (\neg a \cup b))$.

Observe that there are other counter-models. For example, v such that $v(a) = v(b) = \perp_2$ is also a counter model, as $v^*((a \Rightarrow b) \Rightarrow (\neg a \cup b)) = ((\perp_2 \Rightarrow \perp_2) \Rightarrow (\neg\perp_2 \cup \perp_2)) = (T \Rightarrow (\perp_2 \cup \perp_2)) = (T \Rightarrow \perp_2) = \perp_2$.

3. Prove that the equivalence defining $\cup$ in terms of negation and implication in classical logic **does not hold** under $\mathbf{L}_4$, i.e. prove that $(A \cup B) \not\equiv_{\mathbf{L}_4} (\neg A \Rightarrow B)$.

Solution: any v such that $v^*(A) = \perp_2$ and $v^*(B) = \perp_1$ is a **counter- model**. This is not the only counter-model.