CSE/MAT371 QUIZ 5 SOLUTIONS Fall 2016

QUESTION 1 We define, for $A, b_1, b_2, ..., b_n$ and truth assignment v a corresponding formulas A', $B_1, B_2, ..., B_n$ as follows: $A' = \begin{cases} A & \text{if } v^*(A) = T \\ \neg A & \text{if } v^*(A) = F \end{cases}$ $\quad B_i = \begin{cases} b_i & \text{if } v(b_i) = T \\ \neg b_i & \text{if } v(b_i) = F \end{cases}$

We proved the following **Main Lemma**: For any formula $A = A(b_1, b_2, ..., b_n)$ and any truth assignment v, if $A^{'}$, B_1 , B_2, ..., B_n are corresponding formulas defined above, then $B_1, B_2, ..., B_n \vdash A'$.

Let A be a formula $((\neg a \Rightarrow \neg b) \Rightarrow (b \Rightarrow a))$, and let v be such that v(a)= T, v(b) = F.

Write what **Main Lemma** asserts for the formula A.

Solution Observe that the formula A is a basic tautology, hence A' =A.

A = A(a, b) and we get $B_1 = a$, $B_2 = \neg b$ and **Main Lemma** asserts

$$a, \neg b \vdash ((\neg a \Rightarrow \neg b) \Rightarrow (b \Rightarrow a)).$$

QUESTION 2 Consider the Hilbert system $H_2 = (\mathcal{L}_{\{\Rightarrow\}}, \mathcal{F}, \{A1, A2\}, (MP) \frac{A \ ; \ (A \Rightarrow B)}{B})$, where

$A1$: $(A \Rightarrow (B \Rightarrow A))$, $A2$: $((A \Rightarrow (B \Rightarrow C)) \Rightarrow ((A \Rightarrow B) \Rightarrow (A \Rightarrow C)))$,

A3: $((\neg B \Rightarrow \neg A) \Rightarrow ((\neg B \Rightarrow A) \Rightarrow B)))$.

We know that the **Main Lemma** and following formula: $(*) : ((A \Rightarrow B) \Rightarrow ((\neg A \Rightarrow B) \Rightarrow B))$ are **provable** in H_2.

Part 1. Explain why the **Deduction Theorem** holds for H_2.

Solution Only axioms A1 and A2 were used in the proof of the **Deduction Theorem**, so its proof is valid in H_2.

Part 2. The proof of **Completeness Theorem** defines a **method** of efficiently combining $v \in V_A$ while **constructing** the proof of a formula A.

Write all steps of the **Proof 1** as applied to $A : ((\neg a \Rightarrow \neg b) \Rightarrow (b \Rightarrow a))$.

Long Solution The completeness part of the **Completeness Theorem** states: if $\models A$, then $\vdash_{H_2} A$.

The formula $A : ((\neg a \Rightarrow \neg b) \Rightarrow (b \Rightarrow a))$ is a tautology, so the assumption $\models A$ holds and we CAN follow the proof. Also by **Part 1.** the Deduction theorem holds for H_2 and we know that the **Main Lemma** also holds and the formula: $(*)$: $((A \Rightarrow B) \Rightarrow ((\neg A \Rightarrow B) \Rightarrow B))$ is **provable** in H_2.

By the **Main Lemma** and the assumption that $\models A(a, b)$ **any** $v \in V_A$ **defines** formulas B_a, B_b such that

$$B_a, B_b \vdash A.$$

The proof is based on a method of using all $v \in V_A$ (there is 4 of them) to carry a process of **elimination** of the hypothesis B_a, B_b that constructs a formal proof of A, i.e. to prove that $\vdash A$.

Elimination of B_b.

We have to cases: $v(b) = T$ or $v(b) = F$.

Let $v(b) = T$, by definition $B_b = b$ and by the **Main Lemma**, B_a, $b \vdash_{H_2} A$, by **Deduction Theorem** we get $B_a \vdash_{H_2} (b \Rightarrow A$.

Let $v(b) = F$, by definition, $B_b = \neg b$ and by the **Main Lemma**, B_a, $\neg b \vdash_{H_2} A$, by **Deduction Theorem** we get $B_a \vdash_{H_2} (\neg b \Rightarrow A$.

Now we re-write the formula $(*)$ for A= b, B= A and get that $\vdash_{H_2} ((b \Rightarrow A) \Rightarrow ((\neg b \Rightarrow A) \Rightarrow A))$. By monotonicity ,

$$B_a \vdash_{H_2} ((b \Rightarrow A) \Rightarrow ((\neg b \Rightarrow A) \Rightarrow A))$$

we apply MP twice and get $B_a \vdash_{H_2} A$.

We eliminated B_b. We repeat the same process for B_a as follows.

Elimination of B_a.

We have again to cases: $v(a) = T$ or $v(a) = F$.

Let $v(a) = T$, by definition $B_a = a$ and by the **Main Lemma**, $a \vdash_{H_2} A$, by **Deduction Theorem** we get $\vdash_{H_2} (a \Rightarrow A$.

Let $v(a) = F$, by definition, $B_a = \neg a$ and by the **Main Lemma**, $\neg a \vdash_{H_2} A$, by **Deduction Theorem** we get $\vdash_{H_2} (\neg a \Rightarrow A$.

Now we re-write the formula $(*)$ for A= a, B= A and get that $\vdash_{H_2} ((a \Rightarrow A) \Rightarrow ((\neg a \Rightarrow A) \Rightarrow A))$. we apply MP twice and get $\vdash_{H_2} A$.

This **ends the proof**.