CSE/MAT371 QUIZ 4 Fall 2015 Solutions

DEFINITIONS

Write carefully the following DEFINITIONS

D1. Proof System S

By a **proof system** we understand a quadruple

$$S = (\mathcal{L}, \mathcal{E}, LA, \mathcal{R})$$

where

$\mathcal{L} = \{\mathcal{A}, \mathcal{F}\}$ is a **language** of S with a set $\mathcal{F}$ of formulas

$\mathcal{E}$ is a set of **expressions** of S

In particular case $\mathcal{E} = \mathcal{F}$

$LA \subseteq \mathcal{E}$ is a non-empty finite set of **logical axioms** of S

$\mathcal{R}$ is a non-empty **finite set** of **rules of inference** of S

D2. Sound rule of inference (under a semantics **M**)

An inference rule

$$(r) \quad \frac{P_1 \ ; \ P_2 \ ; \ .... \ ; \ P_m}{C}$$

is sound under a semantics **M** if and only if all **M** - models of the set $\{P_1, P_2, .P_m\}$ of its **premisses** are also **M** - models of its **conclusion** C

In particular, in case of **extensional propositional semantics** when the condition below holds for any truth assignment $v : VAR \longrightarrow LV$

If $v \models_{\mathbf{M}} \{P_1, P_2, .P_m\}$, **then** $v \models_{\mathbf{M}} C$

D3 Completeness Theorem for S (under a semantics **M**)

Completeness Theorem (for S and semantics **M**)

$\mathbf{P}_S = \mathbf{T}_{\mathbf{M}}$

i.e. for any $A \in \mathcal{E}$, the following holds: $\vdash_S A$ if and only if $\models_{\mathbf{M}} A$

The **Completeness Theorem** consists of two parts:

Part 1: **Soundness Theorem** : $\mathbf{P}_S \subseteq \mathbf{T}_{\mathbf{M}}$

Part 2: **Completeness Part** of the Completeness Theorem: $\mathbf{T}_{\mathbf{M}} \subseteq \mathbf{P}_S$

PROBLEM

S is the following proof system:

$$S = (\ \mathcal{L}_{\{\Rightarrow, \cup, \neg\}}, \ \mathcal{F}, \ LA = \{(A \Rightarrow (A \cup B))\} \ (r1), \ (r2) \ )$$

Rules of inference:

$$(r1) \ \frac{A \ ; B}{(A \cup \neg B)}, \qquad (r2) \ \frac{A \ ; (A \cup B)}{B}$$

1. Verify whether S is sound/not sound under classical semantics.

Solution The system is not sound. Take any v such that it evaluates $A = T$ and $B = F$. The premiss $(A \cup B$ of the rule $(r2)$ is T and the conclusion B is F

2. Find a formal proof of $\neg(A \Rightarrow (A \cup B))$ in S, i. e. show that $\vdash_S \neg(A \Rightarrow (A \cup B))$

Solution The proof is as follows

B_1**:** $(A \Rightarrow (A \cup B))$, (axiom)

B_2**:** $(A \Rightarrow (A \cup B))$, (axiom)

B_3**:** $((A \Rightarrow (A \cup B)) \cup \neg(A \Rightarrow (A \cup B)))$, (rule r1 application to B_1 and B_2)

B_4**:** $\neg(A \Rightarrow (A \cup B))$, (rule r2 application to B_1 and B_3).

3. Does above point **2.** prove that $\models \neg(A \Rightarrow (A \cup B))$?

Solution

System S is **not sound**, so existence of a proof does not guarantee that what we proved is a tautology.

Moreover, the proof of $\neg(A \Rightarrow (A \cup B))$ used rule $(r2)$ that is not sound!