CSE/MAT371 QUIZ 3 SOLUTIONS Fall 2015

Problem 1

We define a 4 valued $\mathbf{H}_4$ semantics for the language $\mathcal{L} = \mathcal{L}_{\{\neg, \Rightarrow, \cup, \cap\}}$ as follows.

The logical connectives $\neg, \Rightarrow, \cup, \cap$ of $\mathbf{Ł}_4$ are operations in the set $\{F, \bot_1, \bot_2, T\}$, where $F < \bot_1 < \bot_2 < T$ defined as follows.

Conjunction $\cap$ is a function $\cap : \{F, \bot_1, \bot_2, T\} \times \{F, \bot_1, \bot_2, T\} \longrightarrow \{F, \bot_1, \bot_2, , T\}$, such that for any $a, b \in \{F, \bot_1, \bot_2, T\}$, $a \cap b = min\{a, b\}$.

Disjunction $\cup$ is a function $\cup : \{F, \bot_1, \bot_2, T\} \times \{F, \bot_1, \bot_2, T\} \longrightarrow \{F, \bot_1, \bot_2, T\}$, such that for any $a, b \in \{F, \bot_1, \bot_2, T\}$, $a \cup b = max\{a, b\}$.

Implication $\Rightarrow$ is a function $\Rightarrow$: $\{F, \bot_1, \bot_2, T\} \times \{F, \bot_1, \bot_2, T\} \longrightarrow \{F, \bot_1, \bot_2, T\}$, such that for any $a, b \in \{F, \bot_1, \bot_2, T\}$,

$$a \Rightarrow b = \begin{cases} T & \text{if } a \leq b \\ b & \text{otherwise} \end{cases}$$

Negation $\neg a = a \Rightarrow F$

Part 1 Write all Truth Tables for **Implication** and **Negation** in $\mathbf{H}_4$

Solution H_4 **Implication**

$\Rightarrow$	F	$\bot_1$	$\bot_2$	T
F	T	T	T	T
$\bot_1$	F	T	T	T
$\bot_2$	F	$\bot_1$	T	T
T	F	$\bot_1$	$\bot_2$	T

H_4 **Negation**

$\neg$	F	$\bot_1$	$\bot_2$	T
	T	F	F	F

REMEMBER that here the letters a, b represent **LOGICAL VALUES**, as we have by definition that $a, b \in \{F, \bot_1, \bot_2, T\}$

The letters a, b in a **formula** of the language - for example in a formula $((a \Rightarrow b) \Rightarrow (\neg a \cup b))$ - represent **propositional variables** as by **definition** of the formula $a, b \in VAR$

Part 2 Verify whether

$$\models_{\mathbf{H}_4} ((a \Rightarrow b) \Rightarrow (\neg a \cup b))$$

Solution Take any v such that $v(a) = \bot_1$ and $v(b) = \bot_2$.

$v^*((a \Rightarrow b) \Rightarrow (\neg a \cup b)) = ((v^*(a) \Rightarrow v^*(b)) \Rightarrow (\neg v^*(a) \cup v^*(b))) = (\bot_1 \Rightarrow \bot_2) \Rightarrow (\neg \bot_1 \cup \bot_2) = T \Rightarrow (F \cup \bot_2)) = T \Rightarrow \bot_2 = \bot_2$.

This proves that our v is an $\mathbf{H}_4$ **counter-model** and hence

$$\not\models_{\mathbf{H}_4} ((a \Rightarrow b) \Rightarrow (\neg a \cup b))$$

Problem 2

1. Given a formula $A = ((a \cap \neg c) \Rightarrow (\neg a \cup b))$ of a language $\mathcal{L}_{\{\neg,\cap,\cup,\Rightarrow\}}$.

Find a formula B of a language $\mathcal{L}_{\{\neg,\Rightarrow\}}$, such that $A \equiv B$. **List** all proper logical equivalences used at at each step.

Solution :

$$A = ((a \cap \neg c) \Rightarrow (\neg a \cup b)) \equiv ((a \cap \neg c) \Rightarrow (a \Rightarrow b)) \equiv (\neg(a \Rightarrow \neg\neg c) \Rightarrow (a \Rightarrow b)) \equiv (\neg(a \Rightarrow c) \Rightarrow (a \Rightarrow b)) = B$$

Equivalences used:

1. $(\neg A \cup B) \equiv (A \Rightarrow B)$

2. $(A \cap B) \equiv \neg(A \Rightarrow \neg B)$

3. $\neg\neg A \equiv A$

REMEMBER they the logical equivalence symbol $\equiv$ is used **only** in **classical semantic**

For other semantics we use symbol $\equiv_{\mathbf{M}}$ that reads: logical equivalence under a semantics **M**

Problem 3

Prove that $\mathcal{L}_{\{\neg,\cap,\cup,\Rightarrow\}} \equiv \mathcal{L}_{\{\neg,\Rightarrow\}}$

We define the **equivalence of languages** as follows:

Given two languages: $\mathcal{L}_1 = \mathcal{L}_{CON_1}$ and $\mathcal{L}_2 = \mathcal{L}_{CON_2}$, for $CON_1 \neq CON_2$, we say that they are **logically equivalent**, i.e. $\mathcal{L}_1 \equiv \mathcal{L}_2$ if and only if the following conditions **C1, C2** hold.

C1: For every formula A of $\mathcal{L}_1$, there is a formula B of $\mathcal{L}_2$, such that $A \equiv B$,

C2: For every formula C of $\mathcal{L}_2$, there is a formula D of $\mathcal{L}_1$, such that $C \equiv D$.

Solution : We have to prove that $\mathcal{L}_{\{\neg,\Rightarrow\}} \equiv \mathcal{L}_{\{\neg,\cap,\cup,\Rightarrow\}}$.

Condition **C1** holds because $\{\neg, \Rightarrow\} \subseteq \{\neg, \cap, \cup, \Rightarrow\}$.

Condition **C2** holds because of the **Substitution Theorem** and because of the following **logical equivalences** for any for any formulas A, B

$$(A \cap B) \equiv \neg(A \Rightarrow \neg B) \quad \text{and} \quad (A \cup B) \equiv (\neg A \Rightarrow B)$$

Problem 4

Let S be the following set of formulas

$$S = \{(\neg a \Rightarrow (a \cup b)), (a \cup \neg b), (a \Rightarrow (\neg b \Rightarrow a)), ((a \cap b) \Rightarrow b)\}$$

Determine whether S has an $\mathbf{H}_4$ model. Use a shorthand notation.

Solution Observe that connectives for $\mathbf{H}_4$ semantics **restricted** to only $\{T, F\}$ values are identical with definition of **classical connectives**.

Any $v : VAR \rightarrow \{F, \perp_1, \perp_2, T\} \rightarrow$ such that $v(a) = T$, $v(b) = T$ is also a $\mathbf{H}_4$ model for S, by easy evaluation

This is not the only $\mathbf{H}_4$ model!