CSE/MAT371 QUIZ 2 Fall 2015
Solutions

PART 1: DEFINITIONS

All Definitions are for language $\mathcal{L} = \mathcal{L}_{\{\neg, \Rightarrow, \cup, \cap\}}$ and **classical semantics**

D1.

Given the **truth assignment** $v : VAR \longrightarrow \{T, F\}$

Write the **definition** of its **extension** v^* to the set $\mathcal{F}$ of all formulas of $\mathcal{L}$

Definition

We define the **extension** v^* as follows

v^* is a function $v^* : \mathcal{F} \longrightarrow \{T, F\}$ such that

(i) for any $a \in VAR$

$$v^*(a) = v(a)$$

(ii) and for any $A, B \in \mathcal{F}$ we put

$$v^*(\neg A) = \neg v^*(A);$$

$$v^*((A \cap B)) = \cap(v^*(A), v^*(B));$$

$$v^*((A \cup B)) = \cup(v^*(A), v^*(B));$$

$$v^*((A \Rightarrow B)) = \Rightarrow (v^*(A), v^*(B))$$

The condition **(ii)** of the definition of the extension v^* can be also **written** as follows

(ii) and for any $A, B \in \mathcal{F}$ we put

$$v^*(\neg A) = \neg v^*(A);$$

$$v^*((A \cap B)) = v^*(A) \cap v^*(B);$$

$$v^*((A \cup B)) = v^*(A) \cup v^*(B);$$

$$v^*((A \Rightarrow B)) = v^*(A) \Rightarrow v^*(B)$$

D2.

Write the **definition** of a **restricted MODEL** for a given formula $A \in \mathcal{F}$

Definition

A **restricted MODEL** for the formula A is any function $w : VAR_A \longrightarrow \{T, F\}$ such that $w^*(A) = T$, where VAR_A is the sent of all propositional variables appearing in A.

D3

Write the **definition** of a **consistent** non-empty set $\mathcal{G} \subseteq \mathcal{F}$

Definition

A non-empty set $\mathcal{G} \subseteq \mathcal{F}$ of **formulas** is called **consistent** if and only if $\mathcal{G}$ **has a model**, i.e. we have that

$\mathcal{G} \subseteq \mathcal{F}$ is **consistent** if and only if **there is** a truth assignment v such that $v \models \mathcal{G}$,

i.e. v is such that $v^*(A) = T$ for all $A \in \mathcal{G}$

PART 2: PROBLEMS

Problem 1

We know that $v : VAR \longrightarrow \{F, \perp, T\}$ is such that $v^*((a \cap b) \Rightarrow (a \Rightarrow c)) = \perp$

under **H** semantics: for any $(a, b) \in \{T, \perp, F\} \times \{T, \perp, F\}$ we put $a \cup b = max\{a, b\}$, $a \cap b = min\{a, b\}$,

$a \Rightarrow b = \begin{cases} T & \text{if } a \leq b \\ b & \text{otherwise} \end{cases}$ and for any $a \in \{T, \perp, F\}$ we put $\neg a = a \Rightarrow F$

Use above definition and proper reasoning to evaluate $v^*(((b \Rightarrow a) \Rightarrow (a \Rightarrow \neg c)) \cup (a \Rightarrow b))$

Use shorthand notation

Solution

We evaluate

H Implication

$\Rightarrow$	F	$\perp$	T
F	T	T	T
$\perp$	F	T	T
T	F	$\perp$	T

H Negation

$\neg$	F	$\perp$	T
	T	F	F

$v^*((a \cap b) \Rightarrow (a \Rightarrow c)) = \perp$ under **H** semantics if and only if (using a shorthand notation) $(a \cap b) = T$ and $(a \Rightarrow c) = \perp$ if and only if $a = T$, $b = T$ and $(T \Rightarrow c) = \perp$ if and only if $c = \perp$

I.e. we have that $v^*((a \cap b) \Rightarrow (a \Rightarrow c)) = \perp$ if and only if $a = T$, $b = T$, $c = \perp$

Now we can we **evaluate**

$v^*(((b \Rightarrow a) \Rightarrow (a \Rightarrow \neg c)) \cup (a \Rightarrow b))$ as follows

$v^*(((b \Rightarrow a) \Rightarrow (a \Rightarrow \neg c)) \cup (a \Rightarrow b)) = (((T \Rightarrow T) \Rightarrow (T \Rightarrow \neg \perp)) \cup (T \Rightarrow T)) = ((T \Rightarrow (T \Rightarrow F)) \cup T) = T$

Problem 2 (extra 5pts)

Find an **infinite number of formulas** that are **independent** of $\mathcal{G} = \{((a \cap b) \Rightarrow b), (a \cup b), \neg a\}$

Solution

First we have to find all $v : VAR \longrightarrow \{T, F\}$ such that $v \models \{((a \cap b) \Rightarrow b), (a \cup b), \neg a\}$, i.e such that (shorhand notation) $((a \cap b) \Rightarrow b) = T$, $(a \cup b) = T$, $\neg a = T$. Observe that $\models ((a \cap b) \Rightarrow b)$ so we have to consider only $(a \cup b) = T$, $\neg a = T$. This holds if and only if $a = F$ and $(F \cup b) = T$, i.e. if and only if $a = F$ and $b = T$

This proves that $\mathcal{G}$ that any v such that $v(a) = F$ and $v(b) = T$ is the **ONLY one restricted model** for $\mathcal{G}$.

We **define** now a countably in as follows.

The set $VAR - \{a,\ b\}$ is countably infinite , so we take as a set of formulas (to be **proved to be independent**) the set of **atomic formulas**

$$\mathcal{F}_0 = VAR - \{a,\ b\}$$

Let $c \in \mathcal{F}_0$

We define truth assignments $v_1,\ v_2 : VAR \longrightarrow \{T, F\}$ such that

$$v_1 \models \mathcal{G} \cup \{c\} \text{ and } v_2 \models \mathcal{G} \cup \{\neg c\}$$

as follows

$v_1(a) = v(a) = F,\quad v_1(b) = v(b) = T$ and $v_1(c) = T$ **for all** $c \in \mathcal{F}_0$

$v_2(a) = v(a) = F,\quad v_2(b) = v(b) = T$ and $v_2(c) = F$ f**for all** $c \in \mathcal{F}_0$