Chapter 13:
Modal Logics: S4 and S5

Modal logics were first developed, as was the intuitionistic logic, in a form of proof systems only.

First Hilbert style modal proof system was published by Lewis and Langford in 1932.

They presented a formalization for two modal logics, which they called S1 and S2. They also outlined three other proof systems, called S3, S4, and S5.

In 1933 Gödel worked with Heyting's "sentential logic" proof system, what we are calling now Intuitionistic logic.

He considered a particular modal proof system, now known as S4, and asserted that theorems of Heyting's "sentential logic" could be obtained from it by using a certain translation.

Since then hundreds of modal logics have been created.

Some standard texts in the subject are, between the others:

Hughes, Cresswell [1968] for philosophical motivation for various modal and Intuitionistic logic,

Bowen [1979] for a detailed and uniform study of Kripke models for modal logics,

Segeberg [1971] for excellent classification, and

Fitting [1983], for extended and uniform studies of automated proof methods for classes of modal logics.

Godel S4 and S5 Systems

Language

$$\mathcal{L}= \mathcal{L}_{\{\cup,\cap,\Rightarrow,\neg,\mathbf{I},\mathbf{C}\}}$$

The language is common to all modal logics.

Modal logics differ on a choice of axioms and rules of inference, when studied as proof systems and on a choice of semantics when studied semantically.

Axioms: modal logics extend the classical logic hence any modal logic contains two groups of axioms: classical and modal.

Group one: classical axioms Any modal logic adopts as its classical axioms any complete set of axioms for a classical propositional logic.

Group two: modal axioms for S4 and S5

M1 $(\mathbf{I}A \Rightarrow A)$,

M2 $(\mathbf{I}(A \Rightarrow B) \Rightarrow (\mathbf{I}A \Rightarrow \mathbf{I}B))$,

M3 $(\mathbf{I}A \Rightarrow \mathbf{II}A)$,

M4 $(\mathbf{C}A \Rightarrow \mathbf{IC}A)$.

Rules of inference: Modus Ponens (MP) and an additional rule, introduced by Gödel

$$(I)\ \frac{A}{\mathbf{I}A}$$

referred to as *necessitation*.

We define modal logics S4 and S5 as follows.

S4 $= (\ \mathcal{L},\ \mathcal{F},\ \textbf{classical axioms},$
$\mathbf{M1} - \mathbf{M3},\ (MP),\ (I)\ ),$

S5 $= (\ \mathcal{L},\ \mathcal{F},\ \textbf{classical axioms},$
$\mathbf{M1} - \mathbf{M4},\ (MP),\ (I)\ ).$

Observe that the axioms of **S5** extend the axioms of **S4** and both system share the same inference rules, hence we have immediately the following.

FACT For any formula $A \in \mathcal{F}$,

$$if \ \ \vdash_{\mathbf{S4}} A, then \ \ \vdash_{\mathbf{S5}} A.$$

Rasiowa S4 and S5 Systems (1964) It stresses the connection between S4 and S5 and topological spaces which constitute a model for them.

Language uses only one modal connective connective $\mathbf{I}$, that corresponds to the symbol denoting a topological interior of a set.

Language

$$\mathcal{L} = \mathcal{L}_{\{\cap,\cup,\Rightarrow,\neg,\mathbf{I}\}}.$$

Expressibility of connectives

$$\mathbf{I}A \equiv \neg\mathbf{C}\neg A,$$

$$\mathbf{C}A \equiv \neg\mathbf{I}\neg A.$$

Axioms There are, as before, two groups of axioms: classical and modal.

Group one: classical axioms We adopt as classical axioms any complete set of axioms for a classical propositional logic.

Group two: modal axioms for S4 and S5.

I1 $((\mathbf{I}A \cap \mathbf{I}B) \Rightarrow \mathbf{I}(A \cap B))$,

I2 $(\mathbf{I}A \Rightarrow A)$,

I3 $(\mathbf{I}A \Rightarrow \mathbf{II}A)$,

I4 $\mathbf{I}(A \cup \neg A)$,

I5 $(\neg\mathbf{I}\neg A \Rightarrow \mathbf{I}\neg\mathbf{I}\neg A)$

Rules of inference We adopt the Modus Ponens (MP) and an additional modal rule $(\mathbf{I})$,

$$(\mathbf{I}) \ \frac{(A \Rightarrow B)}{(\mathbf{I}A \Rightarrow \mathbf{I}B)}.$$

We define, after Rasiowa, the modal logic proof systems **S4**, **S5** as follows.

$$\mathbf{RS4} = (\ \mathcal{L},\ \mathcal{F},\ \textbf{classical axioms},$$

$$\mathbf{I1} - \mathbf{I4},\ (MP),\ (\mathbf{I}))$$

$$\mathbf{RS4} = (\ \mathcal{L},\ \mathcal{F},\ \textbf{classical axioms},$$

$$\mathbf{I1} - \mathbf{I5},\ (MP),\ (\mathbf{I}))$$

Completeness Theorem For any formula A,

$$\vdash_M A \quad if\ and\ only\ if \quad \models_{S4,S5} A,$$

where M= **S4, S5, RS4, RS5**, respectively.

S4 derivable disjunction (McKinsey, Tarski, 1948)

Let S4 denote any complete system under S4 modal semantics.

$$\vdash_{S4} (IA \cup IB) \quad \textit{if and only if} \quad \vdash_{S4} A \ \textit{ or } \vdash_{S4} B.$$

By the Completeness Theorem we get a general and proof system independent version of the above theorem.

$$\models_{S4} (IA \cup IB) \ \textit{if and only if} \ \models_{S4} \textit{ or } \models_{S4} B.$$

S4 and Intuitionistic logic Gödel was the first to consider the connection between the intuitionistic logic and a logic which was named later S4.

His proof was purely syntactic in its nature, as semantics for neither intuitionistic logic nor modal logic S4 had not been invented yet.

The algebraic proof of this fact, was first published by McKinsey and Tarski in [1948].

Let $\mathcal{L}$ be a propositional language of modal logic

$$\mathcal{L} = \mathcal{L}_{\{\cap,\cup,\Rightarrow,\neg,\mathbf{I}\}},$$

$$\mathcal{L}_0 = \mathcal{L}_{\{\cap,\cup,\Rightarrow,\sim\}}$$

We define, after McKInsey and Tarski (1948) the mapping f as follows.

$$f : \mathcal{F}_0 \quad \rightarrow \mathcal{F}$$

$$fa = \mathbf{Ia} \quad \textit{for any} \quad a \in VAR,$$

$$f(A \Rightarrow B) = \mathbf{I}(fA \Rightarrow fB),$$

$$f(A \cup B) = (fA \cup fB),$$

$$f(A \cap B) = (fA \cap fB),$$

$$f(\sim A) = \mathbf{I}\neg fA,$$

where A, B denote any formulas in $\mathcal{L}_0$.

Example Let A be a formula

$$((\sim A \cap \sim B) \Rightarrow \sim (A \cup B))$$

We evaluate $f(A)$ as follows

$$f((\sim A \cap \sim B) \Rightarrow \sim (A \cup B)) =$$

$$\mathbf{I}(f(\sim A \cap \sim B) \Rightarrow f(\sim (A \cup B)) =$$

$$\mathbf{I}((f(\sim A) \cap f(\sim B)) \Rightarrow f(\sim (A \cup B)) =$$

$$\mathbf{I}((\mathbf{I}\neg fA \cap \mathbf{I}\neg fB) \Rightarrow \mathbf{I}\neg f(A \cup B)) =$$

$$\mathbf{I}((\mathbf{I}\neg A \cap \mathbf{I}\neg B) \Rightarrow \mathbf{I}\neg(fA \cup fB)) =$$

$$\mathbf{I}((\mathbf{I}\neg A \cap \mathbf{I}\neg B) \Rightarrow \mathbf{I}\neg(A \cup B)).$$

Theorem For any formula $A \in \mathcal{F}_0$ of $\mathcal{L}_0$,

$$\vdash_I A, \quad if\ and\ only\ if\ \vdash_{S4} fA,$$

where I, $S4$ denote any complete proof systems under intuitionistic and S4 semantics, respectively.

Theorem For any formula A of $\mathcal{L}_0$,

$$\models_I A, \quad if\ and\ only\ if\ \models_{S4} fA.$$

An embedding of S5 into S4

Embedding theorem 1 For any formula A,

$$\models_{S4} A \quad \textit{if and only if} \quad \models_{S5} \mathbf{IC}A.$$

Embedding theorem 2 For any formula A,

$$\models_{S5} A \quad \textit{if and only if} \quad \models_{S4} \mathbf{ICI}A.$$

Embedding theorem 3 For any formula A,

$$\textit{if} \quad \models_{S5} A, \quad \textit{then} \quad \models_{S4} \neg\mathbf{I}\neg A.$$

The fist proof of the above embedding theorems was given by Matsumoto in 1955.