CHAPTER 11

ch11

Classical Formal Theories

Formal theories play crucial role in mathematics and were historically defined for classical predicate (first order logic) and consequently for other first and higher order logics, classical and non-classical.

The idea of formalism in mathematics, which resulted in the concept of formal theories, or formalized theories, as they are also called. Their concept was developed in connection with the *Hilbert Program*. One of the main objects of the program was to construct a formal theory that would cover the whole mathematics and to prove its *consistency* by employing the simplest of logical means. This part of the program was called the *Consistency Program*, where a formal theory is said to be *consistent* if no formal proof can be carried in that theory for a formula A and at the same time for its negation $\neg A$.

In 1930, while still in his twenties Kurt Gödel made a historic announcement: Hilbert Consistency Program could not be carried out. He justified his claim by proving his Inconsistency Theorem, called also Second Incompleteness Theorem. Roughly speaking the theorem states that a proof of the consistency of every formal theory that contains arithmetic of natural numbers can be carried out only in mathematical theory which is more comprehensive than the one whose consistency is to be proved. In particular, a proof of the consistency of formal (elementary, first order) arithmetic can be carried out only in mathematical theory which contains the whole arithmetic and also other theorems that do not belong to arithmetic. It applies to a formal theory that would cover the whole mathematics because it would obviously contain the arithmetic on natural numbers. Hence the *Hilbert Consistency Program* fails.

Gödel result concerning the proofs of the consistency of formal theories mathematical theories has had a decisive impact on research in properties of formal theories. Instead of looking for direct proofs of inconsistency of mathematical theories mathematicians concentrated largely to relative proofs that demonstrate that a theory under consideration is consistent if a certain other theory, for example a formal theory of natural numbers is consistent. All those proofs are rooted in a deep conviction, even though cannot be proved that the theory of natural numbers is free of inconsistencies. This conviction is confirmed by centuries of development of mathematics and experiences of mathematicians.

A formal theory is called *complete* if for every sentence (formula without free variables) of the language of that theory there is a formal proof of it or of its negation. A formal theory which does not have this property is called *incomplete*. Hence a formal theory is *incomplete* if there is a sentence A of the language of that theory, such that neither A nor $\neg A$ are provable in it. Such sentences are called *undecidable* in the theory in question or *independent* of the theory.

It might seem that one should be able to formalize a theory such as the formal theory of natural numbers is a way to make it complete, i.e. free of undecidable (independent) sentences. But it is not the case in view of Gödel Incompleteness Theorem. It states that every consistent formal theory which contains the arithmetic of natural numbers is incomplete. The Inconsistency Theorem follows form it. This is why the Incompleteness and Inconsistency Theorems are now called Gödel First Incompleteness Theorem 3, Theorem 6 and Gödel Second Incompleteness Theorem 4, Theorem 7, respectively.

The third part of the *Hilbert Program* posed and was concerned with the problem of *decidability* of formal mathematical theories. A formal theory is called **decidable** if there is a method of determining, in a finite number of steps, whether any given formula in that theory is its theorem or not. If a theory is decidable and if the decision algorithm is known, then the study of problems expressible in the language of the theory reduces to a purely mechanical procedure. In undecidable theories there is no mechanical procedure. Most of mathematical theories are undecidable. Gödel proved in 1931 that the arithmetic of of natural numbers is undecidable.

We discuss the *Hilbert Program* and Gödel' s Theorems in more details in sections 3.1 and 3.2, respectively.

1 Formal Theories: Definition and Examples

sec:theory

We define here a notion of a formal theory based on a predicate (first order) language. Formal theories are also routinely called *first order theories, elementary theories, formal axiomatic theories*, or just *theories*, when it is clear from the context that they are formal theories. We will often use the term *theory* for simplicity.

classicTh **Remark 1**

We consider here only classical formal theories, it means theories based and a complete classical Hilbert style proof system. We call it for short a **classical** *Hilbert style proof system. We also assume, for simplicity, that its language contains the full set* $\{\neg, \cap, \cup, \Rightarrow\}$ *of propositional connectives.*

Given a **classical** Hilbert style proof system

$$H = (\mathcal{L},\ \mathcal{F},\ LA,\ \mathcal{R}) \tag{1}$$ Psys

with a predicate (first order) language

$$\mathcal{L} = \mathcal{L}_{\{\neg,\cap,\cup,\Rightarrow\}}(\mathbf{P}, \mathbf{F}, \mathbf{C}),$$

where the sets **P, F, C** are infinitely enumerable.

A formal theory *based on* H is a proof system build from H by adding a new special set SA of axioms to it, called the set of *specific axioms*. They are characteristic description of the universe the formal theory is built to formally describe. This is why we call them specific axioms and denote by SA. The specific axioms are to be true only in a certain structure as opposed to logical axioms LA that are true in all structures.

def:sa **Definition 1 (Specific Axioms)**

Let SA be a certain set of formulas of $\mathcal{L}$ of $H = (\mathcal{L},\ \mathcal{F},\ LA,\ \mathcal{R})$, such that

$$SA \subseteq \mathcal{F} \quad \text{and} \quad LA \cap SA = \emptyset. \tag{2}$$ d:sa

We call the set SA a set of **specific axioms** *of H.*

def:Lsa **Definition 2 (Language $\mathcal{L}_{SA}$)**

Given a proof system $H = (\mathcal{L},\ \mathcal{F},\ LA,\ \mathcal{R})$ and a non-empty set SA (2) of specific axioms. We define a language

$$\mathcal{L}_{SA} \subseteq \mathcal{L} \tag{3}$$ d:Lsa

by restricting the sets **P, F, C** *of predicate, functional, and constant symbols of $\mathcal{L}$ to predicate, functional, and constant symbols appearing in the set SA of specific axioms. Both languages $\mathcal{L}_{SA}$ and $\mathcal{L}$ share the same set of propositional connectives.*

Obviously, if $SA = \emptyset$, then $\mathcal{L}_{SA} = \mathcal{L}$.

Now we are ready to define a formal (first order) classical theory as follows.

def:Th **Definition 3 (Formal Theory)**

A proof system

$$T = (\mathcal{L},\ \mathcal{F},\ LA,\ SA,\ \mathcal{R}), \tag{4}$$ Th

is called a **formal theory** *with the set SA of* **specific axioms**.

The language $\mathcal{L}_{SA}$ defined by (3) is called the **language of the theory** *T.*
The theory T (4) is based on a complete classical proof system

$$H = (\mathcal{L},\ \mathcal{F},\ LA,\ \mathcal{R}).$$

def:thms **Definition 4**

Given a theory $T = (\mathcal{L},\ \mathcal{F},\ LA,\ SA,\ \mathcal{R})$. We denote by $\mathcal{F}_{SA}$ the set of formulas of the language $\mathcal{L}_{SA}$ of T. We denote by $\mathbf{T}$ *the set all provable formulas in the theory T, i.e.*

$$\mathbf{T} = \{B \in \mathcal{F}_{SA} : \quad SA \vdash \ B.\} \tag{5}$$ th:SA

We also write $\vdash_T B$ to denote that $B \in \mathbf{T}$.

d:eq **Definition 5 (Theory with Equality)**

A theory T is called a **theory with equality** *if and only if its language $\mathcal{L}_{SA}$ has as one of its predicates, a two argument predicate P which we denote by $=$, and all Equality Axioms (6) are* **provable** *in T.*

Equality Axioms (6) Eax

For any any free variable or constant of $\mathcal{L}_{SA}$, $R \in \mathbf{P}$, and $t \in \mathbf{T}$, where R is an arbitrary n-ary relation symbol of $\mathcal{L}_{SA}$ and $t \in \mathbf{T}$ is an arbitrary n-ary term of $\mathcal{L}_{SA}$ the following properties hold.

E1 $u = u$,

E2 $(u = w \Rightarrow w = u)$,

E3 $((u_1 = u_2 \cap u_2 = u_3) \Rightarrow u_1 = u_3)$,

E4 $((u_1 = w_1 \cap ... \cap u_n = w_n) \Rightarrow (R(u_1, ..., u_n) \Rightarrow R(w_1, ..., w_n)))$,

E5 $((u_1 = w_1 \cap ... \cap u_n = w_n) \Rightarrow (t(u_1, ..., u_n) \Rightarrow t(w_1, ..., w_n)))$.

Directly from above definitions we have the following.

Fact 1 *The Hilbert style proof system* **H** *defined in chapter* **??** *is a theory with equality with the set of specific axioms $SA = \emptyset$.*

Some Examples of Formal Theories

In practice, formal theories we build and examine their properties are abstract models of real mathematical theories that we develop using freely all laws of logic. Hence the theories we present here are based on a **complete proof system** H for classical predicate logic with a language

$$\mathcal{L} = (\mathcal{L}_{\{\neg, \cap, \cup, \Rightarrow\}}(\mathbf{P}, \mathbf{F}, \mathbf{C}).$$

The first order formal theories are also called *Elementary Theories.*

$T1$. Theory **of equality**

Language

$$\mathcal{L}_{T1} = \mathcal{L}_{\{\neg, \Rightarrow, \cup, \cap\}}(\mathbf{P} = \{P\},\ \mathbf{F} = \emptyset,\ \mathbf{C} = \emptyset),$$

where # P = 2, i.e. P is a two argument predicate. The intended interpretation of P is equality, so we use the equality symbol $=$ instead of P. We write $x = y$ instead $= (x, y)$. We write the language of $T1$ as follows.

$$\mathcal{L}_{T1} = \mathcal{L}_{\{\neg, \Rightarrow, \cup, \cap\}}(\{=\},\ \emptyset,\ \emptyset).$$

Specific Axioms

e1 $\quad x = x$,

e2 $\quad (x = y \Rightarrow y = x)$,

e3 $\quad (x = y \Rightarrow (y = z \Rightarrow x = z))$,

for any $x, y, z \in VAR$,

Exercise 1

Show that the theory $T1$ of equality is a theory with equality of definition 5.

Solution
The first to axioms $e1, e2$ are particular cases of $E1, E2$. We have only to show that the axiom $E3$ is provable in $T1$, i.e. that the formula

$$((x = y \cap y = z) \Rightarrow x = z) \in \mathbf{T1}, \tag{7}$$ e3

where, by (5) $\mathbf{T1} = \{A \in \mathcal{F}_{\{e1,e2,e3\}} : \ \{e1, e2, e3\} \vdash A\}$.

Observe that by definition, $T1$ is based on a complete Hilbert style proof system. A formula

$$(((A \Rightarrow (B \Rightarrow C)) \Rightarrow ((A \cap B) \Rightarrow C))$$

is a predicate tautology, hence is **provable** in $T1$ for any $A, B, C \in \mathcal{F}_{\{e1,e2,e3\}}$. In particular its instance for $A : x = y,\ B : y = z,\ C : x = z$ is also provable in $T1$ what means that

$$(((x = y \Rightarrow (y = z \Rightarrow x = z)) \Rightarrow ((x = y \cap y = z) \Rightarrow x = z)) \in \mathbf{T1}. \tag{8}$$ prov

Applying Modus Ponens (MP) to axiom e3 and (8), we get that

$$((x = y \cap y = z) \Rightarrow x = z) \in \mathbf{T1}.$$

It proves that (7) holds and **ends** the proof.

closed **Observation 1** *We have chosen to write the specific axioms as open formulas. Sometimes it is more convenient to write them as closed formulas (sentences). In this case new axioms will be closures of axioms that were open formulas.*

Taking closures of axioms of $T1$ we obtain the following new formalization for $T1$. We call it $T2$.

$T2$. Theory **of equality** (2)

We adopt a closure of the axioms $e1, e2, e3$, i.e. the following new set of axioms.

Specific Axioms

(e1) $\quad \forall x(x = x)$,

(e2) $\forall x \forall y(x = y \Rightarrow y = x)$,

(e3) $\forall x \forall y \forall z(x = y \Rightarrow (y = z \Rightarrow x = z))$.

$T3$. Theory of **Partial Order**

Partial order relation is also called **order** relation.

Language

$$\mathcal{L}_{T1} = \mathcal{L}_{\{\neg,\Rightarrow,\cup,\cap\}}(\mathbf{P} = \{P, Q\},\ \mathbf{F} = \emptyset,\ \mathbf{C} = \emptyset),$$

where # P = 2, i.e. P is a two argument predicate. The intended interpretation of P is equality, so we use the equality symbol $=$ instead of P. We write $x = y$ instead $= (x, y)$.
Q is a two argument predicate. The intended interpretation of Q is partial order, called also order relation, so we use the order symbol $\leq$ instead of Q. We write $x \leq y$ instead $\leq (x, y)$.
The language is

$$\mathcal{L}_{T3} = \mathcal{L}_{\{\neg,\Rightarrow,\cup,\cap\}}(\{=,\ \leq\},\ \emptyset,\ \emptyset).$$

Specific Axioms

There are two groups of specific axioms: equality and order axioms. We adopt equality axioms (6) to the language $\mathcal{L}_{T3}$ as follows.

Equality Axioms

For any $x, y, z, x_1, x_2, y_1, y_2, \in VAR$,

e1 $x = x$,

e2 $(x = y \Rightarrow y = x)$,

e3 $((x = y \cap y = z) \Rightarrow x = z)$,

e4 $((x_1 = y_1 \cap x_2 = y_2) \Rightarrow (x_1 \leq x_2 \Rightarrow y_1 \leq y_2))$.

Partial Order Axioms

o1 $x \leq x$, *(reflexivity)*

o2 $((x \leq y \cap y \leq x) \Rightarrow x = y)$, *(antisymmetry)*

o3 $((x \leq y \cap y \leq z) \Rightarrow x \leq z)$, *(trasitivity)*

where $x, y, z \in VAR$.

The model of $T3$ is called a *partially ordered structure.*

$T4$. Theory of **Partial Order** (2)

Here is another formalization for partial order.

Language

$$\mathcal{L}_{T4} = \mathcal{L}_{\{\neg,\Rightarrow,\cup,\cap\}}(\mathbf{P} = \{P\},\ \mathbf{F} = \emptyset,\ \mathbf{C} = \emptyset),$$

where # P = 2, i.e. P is a two argument predicate. The intended interpretation of $P(x, y)$ is $x < y$, so we use the "less" symbol $<$ instead of P. We write $x < y$ instead $<(x, y)$. We also write $x \not< y$ for $\neg(x < y)$, i.e. $\neg <(x, y)$.

The language of $T4$ is

$$\mathcal{L}_{T4} = \mathcal{L}_{\{\neg,\Rightarrow,\cup,\cap\}}(\{<\},\ \emptyset,\ \emptyset).$$

Specific Axioms

For any $x, y, z \in VAR$,

p1 $\quad x \not< x,$ $\qquad$ *(irreflexivity)*

p2 $\quad ((x \leq y \cap y \leq z) \Rightarrow x \leq z).$ $\qquad$ *(trasitivity)*

$T5$. **Theory of Linear Order**

Linear order relation is also called **total order** relation.

Language

$$\mathcal{L}_{T5} = \mathcal{L}_{\{\neg,\Rightarrow,\cup,\cap\}}(\{=,\ \leq\},\ \emptyset,\ \emptyset).$$

Specific Axioms

We adopt all axioms of theory $T3$ of partial order and add the following additional axiom.

o4 $\quad (x \leq y) \cup (y \leq x).$

This axiom says that in linearly ordered sets each two elements are comparable.

$T6$. Theory of **Dense Order**

Language

$$\mathcal{L}_{T6} = \mathcal{L}_{\{\neg,\Rightarrow,\cup,\cap\}}(\{=,\ \leq\},\ \emptyset,\ \emptyset).$$

Specific Axioms

We adopt all axioms of theory $T5$ of linear order and add the following additional axiom. We write $x \neq y$ for $\neg(x = y)$, i.e. for the formula $\neg = (x, y)$.

o5 $\quad ((x \leq y \cap x \neq y) \Rightarrow \exists z((x \leq z \cap x \neq z) \cap (z \leq y \cap z \neq y))).$

This axiom says that in linearly ordered sets between any two different there is a third element, respective to the order.

$T7$. **Lattice Theory**

Language

$$\mathcal{L}_{T7} = \mathcal{L}_{\{\neg,\Rightarrow,\cup,\cap\}}(\mathbf{P} = \{P, Q\},\ \mathbf{F} = \{f, g\},\ \mathbf{C} = \emptyset),$$

where # P = 2, i.e. P is a two argument predicate. The intended interpretation of P is equality, so we use the equality symbol $=$ instead of P. We write $x = y$ instead $= (x, y)$.
Q is a two argument predicate. The intended interpretation of Q is partial order, called also order relation, so we use the order symbol $\leq$ instead of Q. We write $x \leq y$ instead $\leq (x, y)$.
f = # g 2, i.e. f, g are a two argument functional symbols. The intended interpretation of f, g is the lattice intersection $\wedge$ and union $\vee$, respectively.
We write $(x \wedge y)$ for $\wedge(x, y)$ and $(x \vee y)$ for $\vee(x, y)$.

Observe that $(x \cap y)$, $(x \cup y)$ are **atomic formulas** of $\mathcal{L}_{T7}$ and $(x \wedge y)$ and $(x \vee y)$ are **terms** of $\mathcal{L}_{T7}$.

We write the language as

$$\mathcal{L}_{T7} = \mathcal{L}_{\{\neg,\Rightarrow,\cup,\cap\}}(\{=, \leq\},\ \{\wedge, \vee\},\ \emptyset).$$

Specific Axioms

There are three groups of specific axioms: equality axioms, order axioms, and lattice axioms. We adopt equality axioms (6) to the language $\mathcal{L}_{T7}$ as follows.

Equality Axioms

For any $x, y, z, x_1, x_2, y_1, y_2, \in VAR$,

e1 $\quad x = x,$

e2 $\quad (x = y \Rightarrow y = x),$

e3 $\quad ((x = y \cap y = z) \Rightarrow x = z),$

e4 $\quad ((x_1 = y_1 \cap x_2 = y_2) \Rightarrow (x_1 \leq x_2 \ \Rightarrow \ y_1 \leq y_2)),$

e5 $\quad ((x_1 = y_1 \cap x_2 = y_2) \Rightarrow (x_1 \wedge x_2 \ \Rightarrow \ y_1 \wedge y_2)),$

e6 $\quad ((x_1 = y_1 \cap x_2 = y_2) \Rightarrow (x_1 \vee x_2 \ \Rightarrow \ y_1 \vee y_2)).$

Remark 2

We write $\wedge$ for the lattice functional symbol of intersection in order to better distinguish it from the conjunction symbol $\cap$ in the formula.
The same applies to the next axiom e7 that involves lattice functional symbol $\vee$ for the union and disjunction symbol $\cup$ in the formula.

Partial Order Axioms

For any $x, y, z \in VAR$,

o1 $\quad x \leq x,$ *(reflexivity)*

o2 $\quad ((x \leq y \cap y \leq x) \Rightarrow x = y),$ *(antisymmetry)*

o3 $\quad ((x \leq y \cap y \leq z) \Rightarrow x \leq z).$ *(trasitivity)*

Lattice Axioms

For any $x, y, z \in VAR$,

b1 $\quad (x \wedge y) = (y \wedge x), \qquad (x \wedge y) = (x \wedge y),$

b2 $\quad (x \wedge (y \wedge z)) = ((x \wedge y) \wedge z), \qquad (x \vee (y \vee z)) = ((x \vee y) \vee z),$

b3 $\quad (((x \wedge y) \vee y) = y), \qquad ((x \wedge (x \vee y)) = x).$

$T8$. Theory of **Distributive Lattices**

Language

$$\mathcal{L}_{T8} = \mathcal{L}_{\{\neg, \Rightarrow, \cup, \cap\}}(\{=, \leq\},\ \{\wedge, \vee\},\ \emptyset).$$

Specific Axioms

We adopt all axioms of theory $T7$ of lattice theory and add the following additional axiom.

b4 $\quad (x \wedge (y \vee z)) = ((x \wedge y) \vee (x \wedge z)).$

$T9$. Theory of **Boolean Algebras**

Language

$$\mathcal{L}_{T9} = \mathcal{L}_{\{\neg, \Rightarrow, \cup, \cap\}}(\{=, \leq\},\ \{\wedge, \vee, -\},\ \emptyset),$$

where $-$ is one argument function symbol representing algebra complement.

Specific Axioms

We adopt all axioms of theory $T8$ of distributive lattices theory and add the following additional axioms that characterize the algebra complement $-$.

b5 $\quad (((x \wedge -x) \vee y) = y), \qquad (((x \vee -x) \wedge y) = y).$

$T10$. Theory of **Groups**

Language

$$\mathcal{L}_{T10} = \mathcal{L}_{\{\neg, \Rightarrow, \cup, \cap\}}(\mathbf{P} = \{P\},\ \mathbf{F} = \{f, g\},\ \mathbf{C} = \{c\}),$$

where # P = 2, i.e. P is a two argument predicate. The intended interpretation of P is equality, so we use the equality symbol $=$ instead of P. We write $x = y$ instead $= (x, y)$.
f is a two argument functional symbol. The intended interpretation of f is group operation $\circ$. We write $(x \circ y)$ for the formula $\circ(x, y)$.
g is a one argument functional symbol. $g(x)$ represent a group inverse element to a given x and we denote it by x^{-1}. We hence use a symbol $^{-1}$ for g.
c is a constant symbol representing unit element in the group and we use a symbol e to denote it

$$\mathcal{L}_{T10} = \mathcal{L}_{\{\neg, \Rightarrow, \cup, \cap\}}(\{=\},\ \{\circ,\ ^{-1}\},\ \{e\}).$$

Specific Axioms

There are two groups of specific axioms: equality axioms and group axioms. We adopt equality axioms (6) to the language $\mathcal{L}_{T10}$ as follows.

Equality Axioms

For any $x, y, z, x_1, x_2, y_1, y_2, \in VAR$,

e1 $\quad x = x$,

e2 $\quad (x = y \Rightarrow y = x)$,

e3 $\quad ((x = y \cap y = z) \Rightarrow x = z)$,

e4 $\quad (x = y \Rightarrow x^{-1} = y^{-1})$,

e5 $\quad ((x_1 = y_1 \cap x_2 = y_2) \Rightarrow (x_1 \circ x_2 \Rightarrow y_1 \circ y_2))$.

Group Axioms

g1 $\quad (x \circ (y \circ z)) = ((x \circ y) \circ z)$,

g2 $\quad (x \circ e) = x$,

g3 $\quad (x \circ x^{-1}) = e$.

$T11$. Theory of **Abelian Groups**

Language is the same as $\mathcal{L}_{T11}$, i.e.

$$\mathcal{L}_{T11} = \mathcal{L}_{\{\neg, \Rightarrow, \cup, \cap\}}(\{=\},\ \{\circ,\ ^{-1}\},\ \{e\})'$$

Specific Axioms

We adopt all axioms of theory $T11$ of groups and add the following additional axiom.

g4 $(x \circ y) = (y \circ x)$.

$T12$. **Theory of Groups** (2)

Here is another formalization for group theory.

Language

$$\mathcal{L}_{T12} = \mathcal{L}_{\{\neg, \Rightarrow, \cup, \cap\}}(\mathbf{P} = \{P\},\ \mathbf{F} = \{f\},\ \mathbf{C} = \{c\}),$$

where # P = 2, i.e. P is a two argument predicate. The intended interpretation of P is equality, so we use the equality symbol $=$ instead of P. We write $x = y$ instead $= (x, y)$.
f is a two argument functional symbols. The intended interpretation of f is group operation $\circ$. We write $(x \circ y)$ for the formula $\circ(x, y)$.
c is a constant symbol representing unit element in the group and we use a unit symbol e to denote it.

$$\mathcal{L}_{T12} = \mathcal{L}_{\{\neg, \Rightarrow, \cup, \cap\}}(\{=\},\ \{\circ\},\ \{e\}).$$

Specific Axioms

For any $x, y, z, x_1, x_2, y_1, y_2, \in VAR$,

a1 $(x \circ (y \circ z)) = ((x \circ y) \circ z)$,

a2 $(x \circ e) = x$,

a3 $\forall x \exists y((x \circ y) = e)$,

a4 $x = x$,

a5 $(x = y \Rightarrow y = x)$,

a6 $(x = y \Rightarrow (y = z \Rightarrow x = z))$,

a7 $(x = y \Rightarrow (x \circ z = y \cap z \circ x = z \circ y))$.

$T13$. Theory of **Abelian Groups** (2)

We adopt the language and all axioms of theory $T12$ of groups and add the following additional axiom.

a8 $(x \circ y) = (y \circ x)$.

Observe that what we formally prove in the formal axiomatic theories presented here represents only fragments of corresponding axiomatic theories developed in mathematics. For example Group Theory, of Boolean Algebras Theory are fields in mathematics and many theorems developed there, like the Representation Theorem for Boolean Algebras, and many, many others in other domains can not be expressed in the languages of respective formal theories. This is is a reason why we also call them elementary theories. We say for example *elementary group theory* to distinguish it from the group theory as a lager field of mathematics.

2 PA: Formal Theory of Natural Numbers

Next to geometry, the theory of natural numbers in the most intuitive and intuitively known of all branches of mathematics. This is why the first attempts to formalize mathematics begin with with arithmetic of natural numbers. The first attempt of axiomatic formalization was given by Dedekind in 1879 and by Peano in 1889. It became known as **Peano Postulates** (axioms) and can be written as follows.

p1 0 is a natural number.

p2 If $n$is a natural number, there is another number which we denote by n'.

We call n' a *successor* of n. The intuitive meaning of n' is $n+1$.

p3 $0 \neq n'$, for any natural number n.

p4 If $n' = m'$, then $n = m$, for any natural numbers n, m.

p5 If W is is a property that may or may not hold for natural numbers, and if (i) 0 has the property W and (ii) whenever a natural number n has the property W, then n' has the property W,
then natural numbers have the property W.

p5 is called *Principle of Induction.*

These axioms together with certain amount of set theory are sufficient to develop not only theory of natural numbers, but also theory of rational and even real numbers. But they can't act as a fully formal theory as they include intuitive notions like "property" and "has a property".

A formal theory of natural numbers based on Peano Postulates is referred in literature as *Peano Arithmetic*, or simply PA. We present here formalization by Mendelson (1973) that is included and worked out in smallest details in his book *Intoduction to Mathematical Logic*(1987). We refer the reader to this excellent book for details and further reading.

We assume, as we did in the previous section **??** that $T14$ and other theories considered here are based on a **complete** Hilbert style proof system

$$H = (\mathcal{L},\ \mathcal{F},\ LA,\ \mathcal{R}) \tag{9}$$ Hsys

for classical predicate logic with a language

$$\mathcal{L} = (\mathcal{L}_{\{\neg,\cap,\cup,\Rightarrow\}}(\mathbf{P}, \mathbf{F}, \mathbf{C}),$$

We additionally assume now that the system H has as one of ts inference rules a a generalization rule

$$(G)\ \frac{A(x)}{\forall x A(x)}. \tag{10}$$ Grule

We do so to facilitate use Mendelson's book as a supplementary reading to the material included here and for additional reading for material not covered here.

dthm **Remark 3**

The Deduction Theorem as proved in chapter **??** *holds for the proof system system H defined by (9).*

$T14.$ **Peano Arithmetic** PA (11) def:PA

Language

$$\mathcal{L}_{PA} = \mathcal{L}(\mathbf{P} = \{P\},\ \mathbf{F} = \{f, g, h\},\ \mathbf{C} = \{c\}),$$

where # P = 2, i.e. P is a two argument predicate. The intended interpretation of P is equality, so we use the equality symbol = instead of P. We write $x = y$ instead $= (x, y)$. We write $x \neq y$ for $\neg(x = y)$.
f is a one argument functional symbol. $f(x)$ represent the successor of a given x and we denote it by x'. We hence use a symbol $'$ for f.
g, h is are two argument functional symbols. The intended interpretation of f is addition and the intended interpretation of g is multiplication. We write $x + y$ for $f(x, y)$ and $x \cdot y$ for $g(x, y)$.

c is a constant symbol representing zero and we use a symbol 0 to denote c.

$$\mathcal{L}_{PA} = \mathcal{L}_{\{\neg,\Rightarrow,\cup,\cap\}}(\{=\},\ \{',\ +,\ \cdot\},\ \{0\}).$$

Specific Axioms

P1 $(x = y \Rightarrow (x = z \Rightarrow y = z))$,

P2 $(x = y \Rightarrow x' = y')$,

P3 $0 \neq x'$,

P4 $(x' = y' \Rightarrow x = y)$,

P5 $x + 0 = x$,

P6 $x + y' = (x + y)'$

P7 $x \cdot 0 = 0$,

P8 $x \cdot y' = (x \cdot y) + x$,

P9 $(A(0) \Rightarrow (\forall x(A(x) \Rightarrow A(x') \Rightarrow \forall x A(x))))$, for all formulas $A(x)$ of $\mathcal{L}_{PA}$ and all $x, y, z \in VAR$.

The axiom P9 is called **Principle of Mathematical Induction**. It does not fully corresponds to Peano Postulate p5 which refers intuitively to all possible properties on natural numbers (uncountably many). The P7 axiom applies only to properties defined by infinitely countably formulas of $A(x)$ of $\mathcal{L}_{PA}$.
Axioms P3, P4 correspond to Peano Postulates p3, p4. The Postulates p1, p2 are taken care by presence of 0 and successor function. Axioms P1, P2 deal with some needed properties of equality that were probably assumed as intuitively obvious by Peano and Dedekind. Axioms P5 - P8 are the recursion equations for addition and multiplication. They are not stated in the Peano Postulates as Dedekind and Peano allowed the use of intuitive set theory within which the existence of addition and multiplication and their property P5-P8 can be proved (Mendelson, 1973).

Observe that while axioms P1 - P6 of theory of Peano Arithmetic PA are particular formulas of $\mathcal{L}_{PA}$, the axiom P9 as an axiom schema providing an infinite number of axioms. This means that the set of axioms P1 - P9 does not provide a *finite axiomatization* for Peano Arithmetic. But any formalization of Peano Postulates must include formalization of the Principle of Induction p5 and hence must contain some form of induction axiom P9. It was proved formally in 1952 by Czeslaw Ryll-Nardzewski and Rabin in 1961.

thm:RN **Theorem 1 (Ryll-Nardzewski)**

Peano Arithmetic is **is not** *finitely axiomatizable; that is there is no theory K having inly a finite number of proper axioms, whose theprems are the same as those of PA.*

By definition 4, given a theory T, we denote by $\mathbf{T}$ the set all provable formulas in T. In particular, **PA** denotes the set of all formulas provable in Peano Arithmetic PA.

Theory PA is one of many formalizations of Peano Arithmetic. They all represent what we call Peano Arithmetic if they have the same set of theorems. We adopt hence the following definition.

d:PA **Definition 6** *Any theory T such that $\mathbf{T} = \mathbf{PA}$ for PA defined by (11) is called a* Peano arithmetic*.*

Taking closure of axioms P1 - P8 of $T14$ we obtain new theory $T15$. The axiom P9 is a sentence (closed formula) already.

T15. Theory CPA

$$\mathcal{L}_{T15} = \mathcal{L}_{T14} = \mathcal{L}_{\{\neg, \Rightarrow, \cup, \cap\}}(\{=\},\ \{',\ +,\ \cdot\},\ \{0\}).$$

We denote the specific axioms of $T15$ by CPA to express that its specific axioms are closures of specific axioms of PA.

Specific Axioms

C1 $\forall x \forall y \forall z(x = y \Rightarrow (x = z \Rightarrow y = z))$,

C2 $\forall x \forall y(x = y \Rightarrow x' = y')$,

C3 $\forall x(0 \neq x')$,

C4 $\forall x \forall y(x' = y' \Rightarrow x = y)$,

C5 $\forall x(x + 0 = x)$,

C6 $\forall x \forall y(x + y' = (x + y)')$

C7 $\forall x(x \cdot 0 = 0)$,

C8 $\forall x \forall y(x \cdot y' = (x \cdot y) + x)$,

C9 $(A(0) \Rightarrow (\forall x(A(x) \Rightarrow A(x')) \Rightarrow \forall x A(x)))$,

for all formulas $A(x)$ of $\mathcal{L}_{PA}$ and all $x, y, z \in VAR$.

Here is a very simple exercise.

f:pa **Fact 2**

Theory CPA is a Peano Arithmetic.

Proof
By definition 6 we have to show that **PA** = **CPA**. Observe that $\mathcal{L}_{CPA} = \mathcal{L}_{PA}$, so we have to show that for any formula B of $\mathcal{L}_{PA}$,

$$\vdash_{PA} B \quad \text{if and only if} \quad \vdash_{CPA} B. \tag{12}$$ CPA

Both theories are based on the same Hilbert proof system H, so to prove (12) means to prove that

(1) all axioms $C1 - C8$ of CPA are provable in PA and vice versa,

(2) all axioms $P1 - P8$ of $\mathcal{L}_{PA}$ are provable in CPA.

Here are detailed proofs for axioms P1, and C1. The proofs for other axioms follow the same pattern.

(1) We prove that the axiom C1 $\forall x \forall y \forall z(x = y \Rightarrow (y = z \Rightarrow x = z))$ is provable in PA as follows.

Observe that axioms of CPA are closures of respective axioms of PA. Consider axiom P1: $(x = y \Rightarrow (y = z \Rightarrow x = z))$. As the proof system H has a generalization rule 10

$$(G)\quad \frac{A(x)}{\forall x A(x)}$$

as its rule of inference, we obtain a proof $B1, B2, B3, B4$ of C1 as follows.

B1: $(x = y \Rightarrow (x = z \Rightarrow y = z))$, (axiom P1)

B2: $\forall z(x = y \Rightarrow (x = z \Rightarrow y = z))$, (GA)

B3: $\forall y \forall z(x = y \Rightarrow (x = z \Rightarrow y = z))$, (GA)

B4: $\forall x \forall y \forall z(x = y \Rightarrow (x = z \Rightarrow y = z))$. (axiom C1)

This **ends the proof** of (1) for axioms P1, and C1.

(2) We prove that the axiom P1 $(x = y \Rightarrow (y = z \Rightarrow x = z))$ of $\mathcal{L}_{PA}$ is provable in CPA as follows.

By H completeness a predicate tautology

$$(\forall x A(x) \Rightarrow A(t)), \tag{13}$$ A4

where term t is free for x in $A(x)$ is provable in H for any formula $A(x)$ of $\mathcal{L}$ and hence for any formula $A(x)$ of its particular sublanguage $\mathcal{L}_{PA}$. So its particular case for $A(x) = (x = y \Rightarrow (x = z \Rightarrow y = z))$ and $t = x$ is provable in CPA, i.e. we have that the formula

$$(\forall x \forall y \forall z(x = y \Rightarrow (x = z \Rightarrow y = z)) \Rightarrow \forall y \forall z(x = y \Rightarrow (x = z \Rightarrow y = z)))$$

is provable in CPA.

We construct a proof $B1, B2, B3, B4, B5, B6, B7$ of P1 in CPA in as follows.

B1 $\forall x \forall y \forall z(x = y \Rightarrow (x = z \Rightarrow y = z))$, (axiom C1)

B2 $(\forall x \forall y \forall z(x = y \Rightarrow (x = z \Rightarrow y = z)) \Rightarrow \forall y \forall z(x = y \Rightarrow (x = z \Rightarrow y = z)))$, by (15)

B3 $\forall y \forall z(x = y \Rightarrow (x = z \Rightarrow y = z))$, MP on B1, B2

B4 $(\forall y \forall z(x = y \Rightarrow (x = z \Rightarrow y = z)) \Rightarrow \forall z(x = y \Rightarrow (x = z \Rightarrow y = z)))$, by (15)

B5 $\forall z(x = y \Rightarrow (x = z \Rightarrow y = z))$, MP on B3, B4

B6 $(\forall z(x = y \Rightarrow (x = z \Rightarrow y = z)) \Rightarrow (x = y \Rightarrow (x = z \Rightarrow y = z)))$, by (15)

B7 $(x = y \Rightarrow (x = z \Rightarrow y = z))$ MP on B5, B6

This **ends the proof** of (2) for axioms P1, and C1.
The proofs for other axioms is similar and are left as homework assignment.

Here are some more basic facts about PA.

f:term **Fact 3**

The following formulas are provable in PA for any terms t, s, r of $\mathcal{L}_{PA}$.

P1' $(t = r \Rightarrow (t = s \Rightarrow r = s))$,

P2' $(t = r \Rightarrow t' = r')$,

P3' $0 \neq t'$,

P4' $(t' = r' \Rightarrow t = r)$,

P5' $t + 0 = t$,

P6' $t + r' = (t + r)'$

P7' $t \cdot 0 = 0$,

P8' $t \cdot r' = (t \cdot r) + t$.

We named the properties as P1'- P8' to stress the fact that they are generalizations of axioms P1 - P8 to the set of all terms of $\mathcal{L}_{PA}$.

Proof

We write the proof for P1' as an example. Proofs of all other formulas follow the same pattern.

Consider axiom P1: $(x = y \Rightarrow (y = z \Rightarrow x = z))$. By Fact 2 its closure $\forall x \forall y \forall z(x = y \Rightarrow (x = z \Rightarrow y = z))$ is provable in Th_{PA}, i.e.

$$\vdash_{PA} \forall x \forall y \forall z(x = y \Rightarrow (x = z \Rightarrow y = z)) \tag{14}$$ a

By H completeness a predicate tautology

$$(\forall x A(x) \Rightarrow A(t)), \tag{15}$$ A4

where term t is free for x in $A(x)$ is provable in H for any formula $A(x)$ of $\mathcal{L}$ and hence for any formula $A(x)$ of its particular sublanguage $\mathcal{L}_{PA}$. So its particular case for $A(x) = \forall y \forall z(x = y \Rightarrow (x = z \Rightarrow y = z))$ the formula (15) is provable in Th_{PA}. Observe that any term t is free for x in this particular $A(x)$. We get that for any term t,

$$\vdash_{PA} (\forall x \forall y \forall z(x = y \Rightarrow (x = z \Rightarrow y = z)) \Rightarrow \forall y \forall z(t = y \Rightarrow (t = z \Rightarrow y = z))), \tag{16}$$ aa

Applying MP to (14) and (16) we get that for any t

$$\vdash_{PA} \forall y \forall z(t = y \Rightarrow (t = z \Rightarrow y = z)). \tag{17}$$ b

Observe that any term r is free for for y in $\forall z(t = y \Rightarrow (t = z \Rightarrow y = z))$. so we have that for all terms r

$$\vdash_{PA} (\forall y \forall z(t = y \Rightarrow (t = z \Rightarrow y = z)) \Rightarrow \forall z(t = r \Rightarrow (t = z \Rightarrow r = z))), \tag{18}$$ bb

as a particular case of 15). Applying MP to (17) and (18) we get that for any terms t, r

$$\vdash_{PA} \forall z(t = r \Rightarrow (t = z \Rightarrow r = z)). \tag{19}$$ c

Observe that any term s is free for for z in $(t = r \Rightarrow (t = z \Rightarrow r = z))$. so we have that

$$\vdash_{PA} (\forall z(t = y \Rightarrow (t = z \Rightarrow y = z)) \Rightarrow (t = r \Rightarrow (t = s \Rightarrow r = s))), \qquad (20)$$ cc

for all terms r, t, s as a particular case of 15). Applying MP to (19) and (20) we get that for any t, r

$$\vdash_{PA} (t = r \Rightarrow (t = s \Rightarrow r = s)).$$

This **ends the proof** of P'.
The proofs of properties P2' - P8' follow the same pattern and are left as a homework assignment.

p3.2 **Fact 4**

The following formulas are provable in PA for any terms t, s, r of $\mathcal{L}_{PA}$.

a1 $t = t$,

a2 $(t = r \Rightarrow r = t)$,

a3 $(t = r \Rightarrow (r = s \Rightarrow t = s))$,

a4 $(r = t \Rightarrow (t = s \Rightarrow r = s))$,

a5 $(t = r \Rightarrow (t + s = r + s))$,

a6 $t = 0 + t$.

Proof
We use in the proof Fact 2, Fact 3, axioms of PA (11, and completeness of the system H. We denote it in the comments. The details of the steps are similar to the proof of Fact 3 and is left to the reader as as homework assignment.

a1 We construct a proof of $t = t$ in CPA in as follows.

B1 $t + 0 = t$, P5' in Fact 3

B2 $(t + 0 = t \Rightarrow (t + 0 = t \Rightarrow t = t))$, P1' in Fact 3 for $t = t + 0, r = t, s = t$

B3 $(t + 0 = t \Rightarrow t = t)$, MP on B1, B2

B4 $t = t$. MP on B1, B3

a2 We construct a proof of $(t = r \Rightarrow r = t)$ as follows.

B1 $(t = r \Rightarrow (t = t \Rightarrow r = t))$, P1' in Fact 3 for $r = t, s = t$

B2 $(t = t \Rightarrow (t = r \Rightarrow r = t))$, B1, tautology

B3 $t = r \Rightarrow r = t$. MP on B2, a1

a3 We construct a proof of $(t = r \Rightarrow (r = s \Rightarrow t = s))$ as follows.

B1 $(r = t \Rightarrow (r = s \Rightarrow t = s))$, P1' in Fact 3

B2 $t = r \Rightarrow r = t$, a2

B3 $(t = r \Rightarrow r = t)$. MP on B1, B2

a4 We construct a proof of $(r = t \Rightarrow (t = s \Rightarrow r = s))$ as follows.

B1 $(r = t \Rightarrow (t = s \Rightarrow r = s))$, a3 for $t = r, r = t$

B2 $(t = s \Rightarrow (r = t \Rightarrow r = s))$, B1, tautology

B3 $s = t \Rightarrow t = s$, a2

B4 $(s = t \Rightarrow (r = t \Rightarrow r = s))$, B1, B2, tautology

B5 $(r = t \Rightarrow (t = s \Rightarrow r = s))$. B4, tautology

a5 We prove $(t = r \Rightarrow (t + s = r + s))$ by the Principle of Mathematical Induction P9 $(A(0) \Rightarrow (\forall x(A(x) \Rightarrow A(x') \Rightarrow \forall x A(x))))$.

The proof uses the Deduction Theorem which holds for the proof system H (Remark 3) and so can be use in PA.

We first apply the Induction Rule to $A(z) : (x = y \Rightarrow x + z = y + z)$ to prove

$$\vdash_{PA} \ \forall z(x = y \Rightarrow x + z = y + z).$$

(i) We prove that $\vdash_{PA} \ A(0)$, i.e. $\vdash_{PA} \ (x = y \Rightarrow x + 0 = y + 0)$. Here the steps in the proof.

B1 $x + 0 = x$, P5'

B2 $y + 0 = y$, P5'

B3 $x = y$, Hyp

B4 $(x + 0 = x \Rightarrow (x = y \Rightarrow x + 0 = y))$, a3 for $t = x + 0, r = x, s = y$

B5 $(x = y \Rightarrow x + 0 = y)$, MP on B1, B4

B6 $x + 0 = y$, MP on B3, B5

B7 $(x+0 = y \Rightarrow (y+0 = y \Rightarrow x+0 = y+0)$, a4 for $r = x+0, t = y, s = y = 0$

B8 $(y + 0 = y \Rightarrow x + 0 = y + 0)$, MP on B6, B7

B9 $x + 0 = y + 0)$, MP on B2, B8

B10 $(x = y \Rightarrow x + 0 = y + 0)$. B1- B9, Deduction Theorem

Thus, $\vdash_{PA} \ A(0)$.

(ii) We prove that $\vdash_{PA} \forall z(A(z) \Rightarrow A(z')$, i.e.

$\forall z((x = y \Rightarrow x + z = y + z) \Rightarrow (x = y \Rightarrow x + z' = y + z'))$. Here the steps in the proof.

C1 $(x = y \Rightarrow x + z = y + z)$, Hyp

C2 $x = y$, Hyp

C3 $x + z' = (x + z)'$, P6'

C4 $y + z' = (y + z)'$, P6'

C5 $x + z = y + z)$, MP on B1, B2

C6 $(x + z = y + z \Rightarrow (x + z)' = (y + z)')$ P2' for $t = x + z, r = y + z$,

C7 $(x + z)' = (y + z)'$, MP on B5, B6

C8 $x + z' = y + z'$, a3 substitution and MP on B3, B7

C9 $((x = y \Rightarrow x + z = y + z) \Rightarrow x + z' = y + z')$ B1- B8, Deduction Theorem

This proves $\vdash A(z) \Rightarrow A(z')$.

C10 $(((x = y \Rightarrow x + 0 = y + 0) \Rightarrow ((x = y \Rightarrow x + z = y + z) \Rightarrow x + z' = y + z')) \Rightarrow \forall z(x = y \Rightarrow x + z = y + z))$, P9 for $A(z) : (x = y \Rightarrow x + z = y + z)$

C11 $((x = y \Rightarrow x + z = y + z) \Rightarrow x + z' = y + z')) \Rightarrow \forall z(x = y \Rightarrow x + z = y + z)$, MP on C10 and B10

C12 $\forall z(x = y \Rightarrow x + z = y + z)$, MP on C11 and C9

C13 $\forall y \forall z(x = y \Rightarrow x + z = y + z)$, (GA)

C14 $\forall x \forall y \forall z(x = y \Rightarrow x + z = y + z)$, (GA)

Now we repeat here the proof of P1' of Fact 3. We apply it step by step to C14. We eliminate the quantifiers $\forall x \forall y \forall z$ and replace variables x, y, z by terms t, r, s using the tautology (15) $(\forall x A(x) \Rightarrow A(t))$ and Modus Ponens. Finally, we obtain the proof of a5, i.e.

$$\vdash_{PA} (t = r \Rightarrow (t + s = r + s)).$$

We go on proving other basic properties of addition and multiplications including for example the following.

f:prop **Fact 5**

The following formulas are provable in PA for any terms t, s, r of $\mathcal{L}_{PA}$.

(i) $t \cdot (r + s) = (t \cdot r) + (t \cdot s)$, *distributivity*

(ii) $(r + s) \cdot t = (r \cdot t) + (s \cdot t)$, *distributivity*

(iii) $(r \cdot t) \cdot s = r \cdot (t \cdot s)$, *associativity of* $\cdot$

(iv) $(t + s = r + s \Rightarrow t = r)$, *canlcellation law for* $+$

Proof

(i) Prove $\vdash_{PA}$ $t \cdot (x+z) = (x \cdot y) + (x \cdot z)$ by induction on z.
(ii) Prove from (i) and property $t \cdot r = r \cdot t$.
(iii) Prove $\vdash_{PA}$ $(x \cdot y) \cdot z = x \cdot (y \cdot z)$ by induction on z.
(iv) Prove $(x + z = y + z \Rightarrow x = y)$ by induction on z

d:num **Definition 7 (Numerals)**

The terms $0,\ 0',\ 0'',\ 0''',\ldots$ *are called* numerals *and denoted by* $\overline{0},\ \overline{1},\ \overline{2},\ \overline{3},\ldots$. *More precisely,*

(1) $\overline{0}$ *is* 0*,*

(2) for any natural number n*,* $\overline{n+1}$ *is* $(\overline{n})'$*.*

In general, if n is a natural number, $\overline{n}$ stands for the corresponding numeral $0'' \ldots '$, i.e. by 0 followed by n strokes.

The numerals can be defined recursively as follows.
(1) 0 is a numeral,
(2) if u is a numeral, then u' is also a numeral.

Here are some more of many properties, intuitively obvious, that provable in Th_{PA}. We give some proofs and an example, and leave the others as an exercise.

Reminder
We use numerals $\overline{n},\ \overline{m}$ as un **abbreviation** of the terms r, s they represent.

Fact 6 *The following formulas are provable in* PA *for any terms* $t,\ s$ *of* $\mathcal{L}_{PA}$*.*

1. $t + \overline{1} = t'$,
2. $t \cdot \overline{1} = t$,
3. $t \cdot \overline{2} = t + t$,
4. $(t + s = 0 \Rightarrow (t = 0 \cap s = 0))$,
5. $(t \neq 0 \Rightarrow (s \cdot t = 0 \Rightarrow s = 0))$,

Proof
1. Major steps in the proof of $t + \overline{1} = t'$ in PA are as follows.
The comments at each step explain how to reconstruct the formal proof from the properties already proven.

B1 $t + 0' = (t+0)'$, P6'

B2 $t + 0 = t$, P5'

B3 $(t+0)' = t'$, B2, P2', MP

B4 $t + 0' = t'$, B1, B3, Fact 4 a3, MP

B5 $t + \overline{1} = t'$. B4, abbreviation

2. Major steps in the proof of $t \cdot \overline{1} = t$ in PA are as follows.

B1 $t \cdot 0' = t \cdot 0 + t$, P8'

B2 $t \cdot 0 = 0$, P7'

B3 $(t \cdot 0) + t = 0 + t$, B1, Fact 4 a4, MP

B4 $t \cdot 0' = 0 + t$, B1, B3, Fact 4 a3, MP

B5 $0 + t = t$, Fact 4 a3, a6, MP

B6 $t \cdot 0' = t$, B4, B5, Fact 4 a3, MP

B7 $t + \overline{1} = t'$, B6, abbreviation

3. Major steps in the proof of $t \cdot \overline{2} = t + t$ in PA are as follows.

B1 $t + \overline{1'} = (t \cdot \overline{1'}) + t$, P8'

B2 $t + \overline{1} = t'$, part 2.

B3 $(t \cdot \overline{1}) + t = t + t$, B2, Fact 4 a5, MP

B4 $t \cdot \overline{1'} = t + t$, B1, B3, Fact 4 a3, MP

B5 $t \cdot \overline{2} = t + t$, B4, abbreviation

4. We prove $(t + s = 0 \Rightarrow (t = 0 \cap s = 0))$ by the following steps.

(s1) We apply the Principle of Mathematical Induction to $A(y) : (x + y = 0 \Rightarrow (x = 0 \cap y = 0))$ and prove

$$\forall y(x + y = 0 \Rightarrow (x = 0 \cap y = 0)). \tag{21}$$ 4ind

(s2) We apply the generalization rule (G) to (**??**) and get

$$\forall x \forall y(x + y = 0 \Rightarrow (x = 0 \cap y = 0)). \tag{22}$$ 4gen

(s3) We repeat here the proof of P1' of Fact 3. We apply it step by step to (22). We eliminate the quantifiers $\forall x \forall y$ and replace variables x, y by terms t, s using the tautology (15) $(\forall x A(x) \Rightarrow A(t))$ and Modus Ponens. Finally, we obtain the proof of 4., i.e.

$$\vdash_{PA} (t + s = 0 \Rightarrow (t = 0 \cap s = 0)).$$

We are going to prove now, as an example, the following.

Fact 7

Let n, m be any natural numbers.

(1) *If $m \neq n$, then $\overline{m} \neq \overline{n}$.*

(2) $\overline{m + n} = \overline{m} + \overline{n}$ *and* $\overline{m \cdot mn} = \overline{m} \cdot \overline{n}$ *are provable in PA.*

(3) *Any model for PA is infinite.*

Proof
Assume $m \neq n$, then $m < n$ or $n < m$. Assume $m < n$. Her are major steps in the formal proof of $\overline{n} \neq \overline{m}$.

The proof uses the Deduction Theorem which holds for the proof system H (Remark 3) and so can be use in PA.

B1 $\overline{m} = \overline{n}$, Hyp

B2 $0'''''' = 0'''''$ B2 is abbreviation of B2 for m applications of $'$ on left side of equation and n applications of $'$ on the right

B3 $0 = t'$, for $t = \overline{n - m - 1}$

We apply P4' m times a=in a row. Then $0 = 0''''$ with $n - m$ applications of $'$ on right side. Let $t = \overline{n - m - 1}$. Since $n > m,\ n - m - 1 \geq 0$. Thus, $0 = t'$.

B4 $0 \neq t'$, P3'

B5 $0 = t' \cap 0 \neq t'$, B3, B4, tautology $(A \Rightarrow (B \Rightarrow (A \cap B)))$, MP

B6 $(\overline{m} = \overline{n} \Rightarrow 0 = t' \cap 0 \neq t')$, B1, B5, Deduction Theorem

B7 $m \neq n$ B6, tautology $((A \Rightarrow (C \cap \neg C)) \Rightarrow \neg A)$, MP

The proof of the case $n < m$ is similar and left to the reader.

(2) We use mathematical induction for natural numbers in the metalanguage with respect to natural number n. Base case. $\overline{m + 0}$ is $\overline{m}$. By P3' $\overline{m} = \overline{m} + \overline{0}$, hence $\overline{m + 0} = \overline{m} + \overline{0}$ and the base step holds.

Inductive step. Assume that $\overline{m + n} = \overline{m} + \overline{n}$ is provable. By P2' and P6' we get $\overline{(m + n)'} = \overline{m} + (\overline{n})'$. But $\overline{m + (n + 1)}$ is $(\overline{m + n})'$ and $\overline{n + 1}$ is $(\overline{n})'$. Hence, $\overline{m + (n + 1)} = \overline{m} + \overline{n + 1}$ and by mathematical induction $\overline{m + n} = \overline{m} + \overline{n}$ is provable in Th_{PA}, for all n, m. The proof that $\overline{m \cdot mn} = \overline{m} \cdot \overline{n}$ is provable in PA for all n, m is similar.

(3) By (2), in a model for PA the objects corresponding to numerals must be distinct. But the set if numerals infinitely countable, so universe of any model for PS must contain infinitely countable subset and hence is infinite.

An order relation can be introduced by in PA as follows.

$\boxed{\texttt{d:ord}}$ **Definition 8 (Order)**

For any terms t, s of $\mathcal{L}_{PA}$, we write

$t < s$ *for a formula* $\exists w(w \neq 0 \cap w + t = s)$,
where we choose w to be the first variable not in t or s,

$t \leq s$ *for a formula* $t < s \cup t = s$,

$t > s$ *for a formula* $s < t$,

$t \geq s$ for a formula $s \leq t$,

$t \not< s$ for a formula $\neg(t < s)$, and so on...

Then we prove properties of order relation, for example the following.

Fact 8

For any terms t, r, s of $\mathcal{L}_{PA}$, the following formulas are provable in PA.

o1 $t \leq t$,

o2 $(t \leq s \Rightarrow (s \leq r \Rightarrow t \leq r))$,

o3 $((t \leq s \cap s \leq t) \Rightarrow t = s)$,

o4 $(t \leq s \Rightarrow (t + r \leq s + r))$,

o5 $(r > 0 \Rightarrow (t > 0 \Rightarrow r \cdot t > 0))$.

There are several stronger forms of the the Principle of Mathematical Induction P9 $(A(0) \Rightarrow (\forall x(A(x) \Rightarrow A(x') \Rightarrow \forall x A(x))))$ that are provable in PA. Here is one of them.

Fact 9 (Complete Induction)

The following formula, called Complete Induction Principle is provable in PA.

$$(\forall x \forall z(z < x \Rightarrow A(z)) \Rightarrow A(x)) \Rightarrow \forall x A(x)).$$

In plain English, Complete Induction Principle says:
consider a property **P** such that , for any x, if **P** holds for for all natural numbers less then x, then **P** holds for x also. Then **P** holds for all natural numbers.

We proved and cited only some of basic properties corresponding to properties of arithmetic of natural numbers. There are many more of them, developed in many Classical Logic textbooks. We refer the reader especially to the Mendelson (1997) book that we found the most rigorous and complete. The proofs included here are more precise and complete versions of few of the Mendelson's proofs.

We selected and proved some direct consequences Peano Arithmetic axioms not only because they are needed as the starting point for a strict development of the the formal theory of arithmetic of natural numbers but also because they are good examples of how one develops any formal theory.

From this point on one can generally translate onto the language $\mathcal{L}_{PA}$ and prove in the PA the results from any text on elementary number theory. Some standard results of number theory are proved with the aid of theory of complex variables and it os often not known whether elementary proofs (or proofs in PA

can be given for such theorems. The statements of some other results of number theory cannot even be formulated in PA.

Hence a natural question about the *strength* and *expressive powers* of PA is a very important one. We will address it shortly in next section with connection of the formulation and proofs of Gödel Theorems. Gödel, in order to prove them developed the whole huge scientific apparatus which grew into new field of Mathematics of *Recursion Theory*, and into another one of *Theory of Computation* with input from Church and Turing.

We know by Ryll Nardzewski Theorem 1 that PA is not finitely axiomatizable. We want to bring reader's attention a finitely axiomatizable proper sub-theory of PA, RR, that has the same expressive power with respect to the Gödel Theorems. Here it is, as formalized and discussed in detail in Mendelson's book.

$T16$. **Robinson System** RR (23) `def:RR`

Language

The language of RR is the same as the language of PA, i.e.

$$\mathcal{L}_{RR} = \mathcal{L}_{\{\neg,\Rightarrow,\cup,\cap\}}(\{=\},\ \{',\ +,\ \cdot\},\ \{0\}).$$

Specific Axioms

r1 $x = x$,

r2 $(x = y \Rightarrow y = x)$,

r3 $(x = y \Rightarrow (y = z \Rightarrow x = z))$,

r4 $(x = y \Rightarrow x' = y')$,

r5 $(x = y \Rightarrow (x + z = y + z \Rightarrow z + x = z + y))$,

r6 $(x = y \Rightarrow (x \cdot z = y \cdot z \Rightarrow z \cdot x = z \cdot y))$,

r7 $(x' = y' \Rightarrow x = y)$,

r8 $0 \neq x'$,

r9 $(x \neq 0 \Rightarrow \exists y\ x = y')$,

r10 $x + 0 = x$,

r11 $x + y' = (x + y)'$,

r12 $x \cdot 0 = 0$,

r13 $x \cdot y' = x \cdot y + x$,

r14 $(y = x \cdot z + p \cap ((p < x \cap y < x \cdot q + r) \cap r < x) \Rightarrow p = r)$.

for any $x, y, z, p, q, r \in VAR$,

Axioms r1 - r13 are due to Robinson (1950), hence the name. Axiom r14 is due to Mendelson (1973). It expresses the uniqueness of remainder. The relation $<$ is as defined by definition 8.

Gödel showed that there are closed formulas of the language $\mathcal{L}_{PA}$ of PA that are neither provable nor disprovable in PA, if PA is consistent. Hence there is a formula that is true under standard interpretation but is not provable in PA. We also see that the *incompleteness* of PA cannot be attributed to omission of some essential axiom but has deeper underlying causes that apply to other theories as well. Robinson proved in 1950, that the Gödel Theorems hold his system RR. In particular RR has the same incompleteness property as PA.

3 Consistency, Completeness, Gödel Theorems

Formal theories because of their precise structure became themselves an object of of mathematical research. The mathematical theory concerned with the study of formalized mathematical theories is called *metamathematics*, the name introduced by Hilbert. The most important open problems of metamathematics introduced by Hilbert as a part of the *Hilbert Program* were concerned with notions of consistency, completeness, and decidability. The answers to Hilbert problems were given by Gödel in1930 in a form of his two theorem that are some of the most important and influential results on twentieth century mathematics. We will discuss here these notions and Gödel's results.

There are two definitions of consistency; semantical and syntactical.

The **semantical** one is based on the notion of a model and says, in plain English: *a theory is consistent if the set of its specific axioms has a model.*

The **syntactical** one uses the notion of provability and says: *a theory is consistent if one can't prove a contradiction in it.*

We have used, in the proof two of the completeness theorem for propositional logic (chapter **??**) the **syntactical** definition of consistency. In chapter **??**, section about the reduction predicate logic to propositional logic we used the **semantical** definition. Both were defined for propositional semantics. We extend now these definitions to the predicate language, predicate semantics, and formal theories. In order to distinguish these two definitions we call the semantic one *model-consistent*, and syntactic one just *consistent*.

th-model **Definition 9 (Model for a Theory)**

Given a first order theory (definition 3)

$$T = (\mathcal{L},\ \mathcal{F},\ LA,\ SA,\ \mathcal{R}).$$

Any structure $\mathcal{M} = [M, I]$ that is a model for the set SA of the specific axioms of T, i.e. such that $\mathcal{M} \models SA$, is called a **model** *for the theory T.*

def:Tcons **Definition 10 (Model - Consistent Theory)**

A first order theory $T = (\mathcal{L},\ \mathcal{F},\ LA,\ SA,\ \mathcal{R})$ is **model - consistent** *if and only if it has a* **model**.

Consider the Peano Arithmetics PA and a structure $\mathcal{M} = [M, I]$ for its language

$$\mathcal{L}_{PA} = \mathcal{L}_{\{\neg,\Rightarrow,\cup,\cap\}}(\{=\},\ \{',\ +,\ \cdot\},\ \{0\}),$$

such that the universe M is the set N of natural numbers (nonnegative integers) and the interpretation I is defined as follows

(1) the constant symbol 0 is interpreted as a natural number 0,

(2) the one argument function symbol $'$ (successor) is interpreted as successor operation (addition of 1) on natural numbers; $succ(n) = n + 1$,

(3) the two argument function symbols $+, \cdot$ are interpreted as ordinary addition and multiplication in N,

(4) the predicate symbol "=" is interpreted as equality relation in N.

def: sM **Definition 11 (Standard Model)**

We denote $\mathcal{M} = [N, I]$ for I defined by (1) - (4) as

$$\mathcal{M} = [N,\ =,\ succ,\ +,\ \cdot\ ] \qquad (24)$$ standM

and call it a **standard model** *for PA. The interpretation I defined by (1) - (4) is called a* **standard interpretation**.

Any model for PA in which the predicate symbol "=" is interpreted as equality relation in N that is not isomorphic to the standard model is called a **nonstandard model** *for PA.*

Observe that if we recognize that the set N of natural numbers with the standard interpretation, i.e. the structure (24) to be a model for PA, then, of course, PA is consistent (model-consistent). However, semantic methods, involving a fair amount of set-theoretic reasoning, are regarded by many (and were regarded as such by Gödel) as too precarious to serve as basis of consistency proofs. Moreover, we have not proved formally that the axioms of PA are true under standard interpretation; we only have taken it as intuitively obvious. Hence for this and other reasons it is common practice to take the model-consistency of PA as un explicit, unproved assumption and to adopt, after Gödel the following syntactic definition of consistency.

def:Tcons **Definition 12 (Consistent Theory)**

Given a theory $T = (\mathcal{L},\ \mathcal{F}\ LA,\ SA,\ \mathcal{R})$.
Let $\mathbf{T}$ *be the set (5) of all provable formulas in* T.

The theory T *is* **consistent** *if and only if* **there is no** *formula* A *of the language* $\mathcal{L}_{SA}$ *such that*

$$\vdash_T\ A \quad and \quad \vdash_T\ \neg A. \tag{25}$$ con

We also write the condition (25) as

$$A \in \mathbf{T} \quad and \quad \neg A \in \mathbf{T}.$$

Directly from definition 12 we get the definition of inconsistency.
We list it separately for its importance to the proof of the Gödel Theorem 4.

def:Tincon **Definition 13 (Inconsistent Theory)**

The theory $T = (\mathcal{L},\ \mathcal{F},\ LA,\ SA,\ \mathcal{R})$ *is* **inconsistent** *if and only if* **there is** *a formula* A *of the language* $\mathcal{L}_{SA}$ *such that*

$$\vdash_T\ A \quad and \quad \vdash_T\ \neg A.$$

Observe that the definitions 12, 13 have purely syntactic meaning. They express the common intuition what proper provability should mean. They say that a provability (formal theory) is a good one (consistent) only when one can't prove a formula and its negation; and is inconsistent when it is possible to prove a contradiction in it.

Here is one of basic characterization of consistent theories.

thm:cons **Theorem 2 (Consistent)**

A theory $T = (\mathcal{L},\ \mathcal{F},\ LA,\ SA,\ \mathcal{R})$ *based on the proof system* $H = (\mathcal{L},\ \mathcal{F},\ LA,\ \mathcal{R})$ *defined by (1) is* **consistent** *if and only if* **there is** *a formula* A *of the language* $\mathcal{L}_{SA}$ *such that*

$$A \notin \mathbf{T}.$$

Proof
Let denote by CC the consistency condition in the definition 12 and by CT consistency condition in the theorem 2.

1. We prove implication " if CC, then CT".
Assume not CT. This means that $A \in \mathbf{T}$ for all formulas A,

$$A \in \mathbf{T} \quad \text{and} \quad \neg A \in \mathbf{T}. \tag{26}$$ c1

In particular there is B such that and $B \in \mathbf{T}$ and $\neg B \in \mathbf{T}$ and not CC holds.

2. We prove implication " if CT, then CD".
Assume not CD. This means that there is A of $\mathcal{L}_{SA}$, such that $A \in \mathbf{T}$. By definition 12 all tautologies are derivable in $\mathbf{T}$. Hence

$$(((A \cap B) \Rightarrow C) \Rightarrow ((A \Rightarrow (B \Rightarrow C)))), \quad ((A \cap \neg A) \Rightarrow C) \in \mathbf{T}, \tag{27}$$ t1

for all $A, B, C \in \mathcal{F}$. In particular, when $B = \neg A$ we get that

$$(((A \cap \neg A) \Rightarrow C) \Rightarrow ((A \Rightarrow (\neg A \Rightarrow C)))) \in \mathbf{T}. \tag{28}$$ t2

Applying MP (27) and (28) we get

$$((A \Rightarrow (\neg A \Rightarrow C))) \in \mathbf{T}. \tag{29}$$ t3

Applying MP twice to (29) and (26) we get that $C \in \mathbf{T}$, for all C. We proved not CT. This **ends the proof** of 2. and of the theorem.

Theorem 2 often serves a following definition of consistency.

d:cons2 **Definition 14**

A theory T is **consistent** *if and only if* $\mathbf{T} \neq \mathcal{F}_{SA}$, *i.e. there is A of $\mathcal{L}_{SA}$, such that $A \notin \mathbf{T}$.*

The next important characterization of a formal theory is the one of its completeness understood as the ability of proving or disapproving any of its statements, provided it is correctly formulated ' in its language.

def: comp **Definition 15 (Complete Theory)**

A theory $T = (\mathcal{L}, \mathcal{F}, LA, SA, \mathcal{R})$ is **complete** *if and only if for any closed formula (sentence) A of the language $\mathcal{L}_{SA}$,*

$$\vdash_T A \quad or \quad \vdash_T \neg A. \tag{30}$$ c2

We also write the condition (30) as

$$A \in \mathbf{T} \quad or \quad \neg A \in \mathbf{T}. \tag{31}$$ c3

Directly from definition 15 we get the definition of incompleteness.
We list it separately for its importance to the proof of the Gödel Incompleteness Theorem 3, Theorem 6.

def:incomp **Definition 16 (Incomplete Theory)**

A theory T is **incomplete** *if and only if there is a closed formula (sentence) A of the language $\mathcal{L}_{SA}$, such that*

$$\nvdash_T A \quad and \quad \nvdash_T \neg A. \tag{32}$$ c4

We also write the condition (32) as

$$A \notin \mathbf{T} \quad \textit{and} \quad \neg A \notin \mathbf{T}.$$

Any sentence A with the property (32) is called an **independent**, *or* **undecidable** *sentence of the theory T.*

By definition 16, in order to prove that a given theory T is incomplete we have to construct a sentence A of $\mathcal{L}_{SA}$ and prove that either A nor $\neg A$ has a proof in it.

We are now almost ready to discuss Gödel Theorems. One of the most (if not the most) comprehensive development with detailed and strict proofs of all what is needed to carry proofs of Gödel Theorems can be found the Mendelson (1984) book. The Gödel Theorems chapter is over 50 pages long, technically sound and beautiful. We are confident that our readers, at this stage of our book are ready and able to follow Mendelson's or other authors work.

We present here a short, high level approach adopting style of Smorynski's chapter in the *Handbook of Mathematical Logic*, Studies in Logic and Foundations of Mathematics, Volume 20 (1977). The chapter also is over 40 pages long (it seems to be a norm when one wants to really prove Gödel's results). It is written in a very condensed and general way and concentrates on presentation of modern results. It assumes that readers are already familiar with the traditional approach so beautifully presented in Mendelson's book, but I encourage readers to reach for it, as it is, in its own style a very interesting work.

We also want to bring to readers attention that the introduction to the Smorynski's chapter contains an excellent discussion of Hilbert Program and its relationship to Gödel's results. It gives an explanation why and how devastating Gödel Theorems were to the optimism reflected in Hilbert's Consistency and Conservation Programs.

3.1 Hilbert's Conservation and Consistency Programs

sec:HP

Hilbert proposed his Conservation Program and Consistency Programs as response to L.E.J. Brouwer and Herman Weyl (1920) believe and propagation of their believes that existence (as early as 1908) of Zermello's paradoxes free axiomatization of set theory makes the need for investigations (and proof) into consistency of mathematics superfluous. Hilbert decided to intervene. He wrote:

" .... they (Brouwer and Weil) would chop and mangle the science. If we would follow such a reform as the one they suggest,we would run the risk of losing a great part of our most valuable treasures!"

Hilbert stated his **Conservation Program** as follows: To justify the use o abstract techniques he would show - by as simple and concrete a means as possible - that the use of abstract techniques was *conservative* - i.e. that any concrete assertion one could derive by means of such abstract techniques would be derivable without them.

We follow Smorynski's clarification of some of Hilbertian jargon whose exact meaning was never defined by Hilbert. We hence talk, in the domain of mathematics, about *finitistically meaningful* statements and *finitistic* means of proof.

By the *finitistically meaningful* statements we mean for example identities of the form

$$\forall x(f(x) = g(x)),$$

where f, g are reasonably simple functions, for example primitive recursive. We will call them *real* statements. *Finitistic* proofs correspond to computations or combinatorial manipulations.

More complicated statements are called *ideal* ones and, as such, have no meaning, but can be manipulated abstractly and the use of ideal statements and abstract reasoning about them would not allow one to derive any new *real* statements, i.e. none which were not already derivable To refute Weyl and Brouwer, Hilbert required that this latter *conservation property* itself be finitistically provable.

Hilbert's **Consistency Program** asks to devise a finitistic means of proving the consistency of various formal systems *encoding* abstract reasoning with *ideal* statements.

The Consistency Program is a natural outgrowth and successor to the Conservation Program. There are two reason reasons for this.
R1. *Consistency* is the assertion that some string of symbols is not provable. Since derivations are simple combinatorial manipulations, this is a *finitistically meaningful* and ought to have a *finitistic proof.*

R2. Proving a consistency of a formal system encoding the abstract concepts already establishes the conservation result!

Reason R1 is straightforward. We will discuss R2 as it is particularly important.

Let's denote by R a formal systems encoding real statements with their finitistic proofs and by I the *ideal* system with its abstract reasoning.

Let A be a *real* statement $\forall x(f(x) = g(x))$.

Assume $\vdash_I A$. Then there is a derivation d of A in I. But, derivations are concrete objects and, for some real formula $P(x, y)$ *encoding* derivations in I,

$$\vdash_R P(d, \ulcorner A \urcorner),$$

where $\ulcorner A \urcorner$ is some *code* for A.

Now, if A were false, one would have $f(a) \neq g(a)$ for some a and hence

$$\vdash_R P(c, \ulcorner \neg A \urcorner)$$

for some c. In fact, one would have a stronger assertion

$$\vdash_R (f(x) \neq g(x) \Rightarrow P(c_x, \ulcorner \neg A \urcorner)).$$

But, if R proves *consistency* of I, we have

$$\vdash_R \neg(P(d, \ulcorner A \urcorner) \cap P(c, \ulcorner \neg A \urcorner)),$$

whence $\vdash_R f(x) = g(x)$, with free variable x, i.e. $\vdash_R \forall x(f(x) = g(x))$.

To make the above argument rigorous, one has to define and explain the basics of *encoding*, the assumptions on the the formula $P(x, y)$ and to deliver the whole argument is a formal rigorous way, i.e. to develop rigorously the whole apparatus developed originally by Gödel and needed for the proofs of his theorems. We bring it here because it clearly invited Hilbert to establish his Consistency Program. Since Consistency Program was as broad as the general Conservation Program and, since it was more tractable, Hilbert fixed on it asserting:

"if the arbitrary given axioms do not contradict each other through their consequences, then they are true, then the objects defined through the axioms exist. That, for me, is the criterion og truth and existence".

The Consistency Program had as its goal the proof, by finitistic means of the consistence of strong systems. The solution would completely justify the use of abstract concepts and would repudiate Brouwer and Weyl.
Gödel proved that it couldn't work.

3.2 Gödel Incompleteness Theorems

sec:GT

In 1920, while in his twenties, Kurt Gödel announced that Hilbert's Consistency Program could not be carried out. He had proved two theorems which gave a blow to the HIlbert's Program but on the other hand changed the face of mathematics establishing mathematical logic as its strong and rapidly developing discipline.

Loosely stated these theorems are:

thm:G1 **Theorem 3 (First Incompleteness Theorem)**

Let T be a formal theory containing arithmetic. Then there is a sentence A in the language of T which asserts its own unprovability and is such that:

(i) If T is consistent, then $\nvdash_T A$.

(ii) If T is ω- consistent, then $\nvdash_T \neg A$.

thm:G2 **Theorem 4 (Second Incompleteness Theorem)**

Let T be a consistent formal theory containing arithmetic. Then

$$\nvdash_T Con_T,$$

where Con_T is the sentence in the language of T asserting the consistency of T.

Observe that the Second Incompleteness Theorem destroys the Consistency Program. It states that R can't prove its own consistency, so obviously it can't prove consistency of I.

Smorynski argument that the First Incompleteness Theorem destroys the Conservation Program is as follows. The the sentence A is real and is easily seen to be true. It asserts its own unprovability and is indeed unprovable. Thus the Conservation Program cannot be carried out and, hence, the same must hold for the Consistency Program.

M.Detlefsen in the Appendix of his book "Hilbert Program: An Essay on Mathematical Instrumentalism", Springer, 2013, argues that Smorynski's argument is ambiguous, as he doesn't tell us whether it is unprovability in R or unprovability in I. We recommend to the reader interested a philosophical discussion of Hilbert Program to read this Appendix, if not the whole book.

We will now formulate the Incompleteness Theorems in a more precise formal way and describe the main ideas behind the their proofs.

Arithmetization and Encoding (33) arithm

Observe that that in order to formalize the Incompleteness Theorems one has first to "translate" the sentences A and Con_T into the language of T. For the First Incompleteness Theorems 3 one needs to to " translate " a self-referring sentence *"I am not provable in a theory T"*; for the Second Theorem 4 the self-referring sentence is *"I am consistent"*.

The assumption in both theorems is that T contains arithmetic means usually it contains the Peano Arithmetic PA (11), or even its sub-theory RR (23), called Robinson System. In this case the final product of such "translation" must be a sentence A or sentence Con_T of the language $\mathcal{L}_{PA}$ of PA, usually written as

$$\mathcal{L}_{PA} = \mathcal{L}(\{=\},\ \{',\ +,\ \cdot\},\ \{0\}).$$

This "translation" process into the language of some formal system containing arithmetic is called *arithmetization* and *encoding*, or *encoding* for short. We define a notion of *arithmetization* as follows.

An arithmetization of a theory T is a one-to-one function g from the set of symbols of the language of T, expressions (formulas) of T, and finite sequences of expressions of T (proofs) into the set of positive integers. The function g must satisfy the following conditions.
(1) g is effectively computable;
(2) there is an effective procedure that determines whether any given positive integer n is in the range of g and, if n is in the range of g, the procedure finds the object x such that $g(x) = m$.

Arithmetization, i.e. a method of associating numbers with symbols, expressions, and sequences of expressions was originally devised by Gödel in 1931 in order to *arithmetize* Peano Arithmetic PA and *encode* the arithmetization process PA in order to formulate and to prove his Incompleteness Theorems 3, 4.

Functions and relations whose arguments and values are natural numbers are called the number-theoretic functions and relations.

In order to arithmetize and encode in a formal system, say PA we have to

1. associate numbers with symbols symbols of the language of the system, associate numbers with expressions, and sequences of expressions of the language of the system (arithmetization, encoding of basic syntax, and encoding of syntax)

2. replace assertions *about* the system by number-theoretic statements, and express these number-theoretic statements *within* the formal system itself (arithmetization,, encoding).

We want the number - theoretic function to be *representable* PA and the predicates to be *expressible* in PA, i.e. their characteristic functions to be *representable* in PA.

The study of *representability* of functions in PA leads to the class of number-theoretic functions that turn out to be of great importance in mathematical logic, namely the x *primitive recursive* and *recursive functions.* Their definition and study in a form of a *Recursion Theory* is an important field of mathematics and of computer science which developed out of the Gödel proof of the Incompleteness Theorems.

We prove that the class of recursive functions is identical with the class of functions representable in PA, i.e. we prove: *every recursive function is representable in PA* and *every function representable in PA is recursive.*

The representability of primitive recursive and recursive functions in S in general and in PA in particular plays crucial role in the encoding process and consequently in the proof of Gödel Theorems.

The details of arithmetization and encoding are as complicated and tedious as fascinating but are out of scope of our book. We recommend again, Mendelson book "Introduction to Mathematical Logic", 4th ed.Chapman&Hall (1997) as the one with the most comprehensive and detailed presentation.

Theories T and S (34) d:TS

We assume at this moment that T is some fixed, but for a moment unspecified **consistent** formal theory. We also assume that **encoding** is done in some fixed theory S and that T contains S, i.e. the language of T is an extension of the language of S and

$$\mathbf{S} \subseteq \mathbf{T},$$

i.e. for any formula A,

$$\text{if } \vdash_S \ A, \text{ then } \ \vdash_T \ A. \tag{35}$$ as:TS

We also assume that T and S contain as constants only numerals (definition 7)

$$\overline{0},\ \overline{1},\ \overline{2},\ \overline{3}, \ldots,$$

and T contains infinitely countably many functional and predicate symbols.

Usually S is taken to be a formal theory of arithmetic, but sometimes S can be a weak set theory. But in any case S always *contains numerals.*

We also assume that theories T and S as defined by (34) are such that the following Principles of Encoding (36) hold.

Principles of Encoding for T and S (36) code

The mechanics, conditions and details of *encoding* for T and S for S being Peano Arithmetic PA or its sub-theory Robinson Arithmetic RR (**??**) are beautifully presented in in the smallest detail in Mendelson book.

The Smorynski's approach we discuss here covers a larger class of formal theories and uses a more general and modern approach. We can't include all details but we are convinced that at this stage the reader will be able to follow Smorynski's chapter in the Encyclopedia. The chapter is very well and clearly written and is now classical. We wholeheartedly recommend it as a future reading.

We also follow Smorynski approach explaining *what* is to be encoded, *where* it is to be encoded, and which are the most important encoding and provability conditions needed for the proofs of the Incompleteness Theorems. We encourage reader to read the chapter for follow details.

We first encode the *syntax* of T in S.
Since encoding takes place in S, it has a sufficient supply of *constants* (countably infinite set of numerals

$$\overline{0},\ \overline{1},\ \overline{2},\ \overline{3}, \ldots,$$

and *closed terms* to be used as codes.

We assign to each formula A of the language of T a *closed term*,

$$\ulcorner A \urcorner$$

called the *code* of A. If $A(x)$ is a formula with a free variable x, then the code $\ulcorner A(x) \urcorner$ is a closed term encoding the formula $A(x)$, with x viewed as a *syntactic object* and not as a parameter.

We do it recursively, first we assign *codes* (unique closed terms from S) to its basic syntactic objects, i.e. elements of the alphabet of the language of T.

Terms and formulas are finite sequences of these symbols and derivations (formal proofs) are finite sequences of formulas. It means that S have to be able to *encode* and *manipulate* finite sequences. We use for such encoding a class primitive recursive functions and relations. We assume S admits a *representation* of these functions and relations and finish encoding syntax.

S will also have to have certain function symbols and we have to be able to encode them.

1. S must have we functional symbols, *neg*, *impl*, etc., corresponding to the logical connectives and quantifiers, such that, such that, for all formulas A, B of the language of T,

$$\vdash_S \; neg(\ulcorner A \urcorner), \quad \vdash_S \; impl(\ulcorner A \Rightarrow B \urcorner), \; etc.$$

An operation of substitution of a variable x in a formula A(x) by a term t is of a special importance in logic, so it must be represented in S, i.e.

2. S must have in a functional symbol *sub* that represents the substitution operator, such that for any formula $A(x)$ and term t with codes $\ulcorner A(x) \urcorner$, $\ulcorner t \urcorner$, respectively,

$$\vdash_S \; sub(\ulcorner A(x) \urcorner, \; \ulcorner t \urcorner) = \ulcorner A(t) \urcorner. \tag{37}$$

subs

Iteratation of *sub* allows one to define $sub_3, \; sub_4, \; sub_5, \ldots$, such that

$$\vdash_S \; sub_n(\ulcorner A(x_1, \ldots, x_n) \urcorner, \; \ulcorner t_1 \urcorner, \ldots, \ulcorner t_n \urcorner) = \ulcorner A(t_1, \ldots, t_n) \urcorner.$$

Finally, we have to encode derivations in S , i.e.

3. S has to have in a binary relation $Prov_T(x, \; y)$, such that for closed terms t_1, t_2,

$\vdash_S \; Prov_T(t_1, \; t_2)$ if and only if t_1 is a code of a derivation in T of the formula with a code t_2.

We read $Prov_T(x, \; y)$ as "x proves y in T " or " x is a proof of y in T".
It follows that for some closed term t,

$$\vdash_T \; A \text{ if and only if } \vdash_S \; Prov_T(t, \; \ulcorner A \urcorner).$$

We define

$$Pr_T(y) \; \Leftrightarrow \; \exists x Prov_T(x, \; y) \tag{38}$$

d:Pr

and obtain a predicate asserting provability.

However, it is not always true

$$\vdash \; T \; A \text{ if and only if } \vdash_S \; Pr_T(\ulcorner A \urcorner),$$

unless S is fairly *sound* (to be defined separately).

The encoding can be carried out, however, in such a way that the following conditions essential to the proofs of the Incompleteness Theorems hold for any sentence A of T.

Derivability Conditions (Hilbert-Bernays, 1939) (39) DC

D1 $\vdash_T\ A$ implies $\vdash_S\ Pr_T(\ulcorner A \urcorner)$.

D2 $\vdash_S\ ((Pr_T(\ulcorner A \urcorner) \Rightarrow Pr_T(\ulcorner Pr_T(\ulcorner A \urcorner)\urcorner)))$.

D3 $\vdash_S\ ((Pr_T(\ulcorner A \urcorner) \cap Pr_T(\ulcorner (A \Rightarrow B)\urcorner)) \Rightarrow Pr_T(\ulcorner B \urcorner))$.

Proof of the Incompleteness Theorems (40) proof

The following theorem 5 is essential to the proof of the Incompleteness Theorems. It is called historically *Diagonalization Lemma* or *Fixed Point Theorem* and both names are used interchangeably. The fist name as is historically older, important for convenience of references and the second name is routinely used in computer science community.

Mendelson (1977) believes that the central idea was first explicitly mentions by Carnap who pointed out in 1934 that the result was implicit in the work of Gödel (1931). Gödel was not aware of Carnap work until 1937.

The theorem 5 is called Diagonalization Lemma because the argument used in its proof has some resemblance to the the *diagonal arguments* used by Cantor in 1891. He first used it proving that there are infinite sets that can not be put in one-to-one correspondence with the set on natural numbers. He then used its generalization in the proof of his famous Cantor Theorem: for every set X, its set of all subsets has a larger cardinality than X itself (see chapter **??**).

In mathematics, a *fixed-point theorem* is a name of a theorem saying that a function f under some conditions, will have a at least one fixed point, i.e. a point x such that $f(x) = x$.

The theorem 5 says that for any formula A in the language of theory T with one free variable there is a sentence B such that the formula $(B \Leftrightarrow A(\ulcorner B \urcorner))$ is provable in T.

Intuitively, B is a *self-referential* sentence saying that B has property A. The sentence B can be viewed as a *fixed point* of the operation assigning to each formula A the sentence $A(\ulcorner B \urcorner)$. Hence the name *Fixed Point Theorem.*

Theorem 5 proves the existence of self-referential sentences in certain formal theories of natural numbers. These sentences then, in turn, are to be used to prove Gödel's Incompleteness Theorems. Here it is.

thm:diag **Theorem 5 (Diagonalization Lemma)**

Let T, S be theories defined by (34).
Let $A(x)$ be a formula in the language of T with x as the only free variable. Then there is a sentence B such that

$$\vdash_S \ (B \Leftrightarrow A(\ulcorner B \urcorner)).$$

NOTE: If A, B are not in the language of S, then by $\vdash_S \ (B \Leftrightarrow A(\ulcorner B \urcorner))$ we mean that the equivalence is proved in the theoryS' in the language of T whose only non-logical axioms are those of S.

Proof
Given $A(x)$, let $(C(x) \Leftrightarrow A(sub(x,x))$ be a diagonalization of $A(x)$.

Let $m = \ulcorner C(x) \urcorner$ and $B = C(m)$.

Then we claim

$$\vdash_S \ (B \Leftrightarrow A(\ulcorner B \urcorner)).$$

For, in S, we see that

$$\begin{aligned} B \ &\Leftrightarrow C(m) \Leftrightarrow A(sub(m,m)) \\ &\Leftrightarrow A(sub(\ulcorner C(x) \urcorner, \ m) \qquad \text{(since } m = \ulcorner C(x) \urcorner) \\ &\Leftrightarrow A(\ulcorner C(m) \urcorner) \Leftrightarrow A(\ulcorner B \urcorner) \qquad \text{by (37)) and } B = C(m). \end{aligned}$$

This proves (we leave details to the reader as a homework exercise)

$$\vdash_S \ (B \Leftrightarrow A(\ulcorner B \urcorner)).$$

thm:Inc1 **Theorem 6 (First Incompleteness Theorem)**

Let T, S be theories defined by (34).
Then there is a sentence G in the language of T such that:

(i) $\nvdash_T \ G$.

(ii) *under an additional assumption,* $\nvdash_T \ \neg A$.

Proof
Applying Diagonalization Lemma 5 for a formula $A(x)$ being $\neg Pr_T(x)$, where $Pr_T(x)$ is defined by (38) we get that there is a sentence G such that

$$\vdash_S \ (G \Leftrightarrow \ \neg Pr_T(\ulcorner G \urcorner)).$$

By the assumed property (35) in the definition (34) of T, S we have that also

$$\vdash_T \ (G \Leftrightarrow \ \neg Pr_T(\ulcorner G \urcorner)). \tag{41}$$ prop

(i) We conduct the proof by contradiction. Assume $\vdash_T\ G$.
Observe that $\vdash_T\ G$ implies $\vdash_T\ Pr_T(\ulcorner G \urcorner)$ by **D1** and (35). This and the above (41) contradicts the *consistency* of T.

(ii) The *additional assumption* is assuming that the converse implication to **D1** holds, i.e that $\vdash_T\ Pr_T(\ulcorner G \urcorner)$ implies $\vdash_T\ G$.

We conduct the proof by contradiction. Assume $\vdash_T\ \neg G$.
Hence $\vdash_T\ \neg\neg Pr_T(\ulcorner B \urcorner))$ so we have that $\vdash_T\ Pr_T(\ulcorner B \urcorner))$. By the *additional assumption* it implies that $\vdash_T\ G$ what contradicts contradicting the consistency of T.
This **ends** the proof.

Observe that the sentence G is equivalent in T to an assertion that G is unprovable in T. In other words it says *" I am not provable in T"* and hence theorem 6 is a strict mathematical formalization of the intuitively stated theorem 3. We call G the Gödel's sentence.

thm:Inc2 **Theorem 7 (Second Incompleteness Theorem)**

Let T, S be theories defined by (34).
Let Con_T be a sentence $\neg Pr_T(\ulcorner C \urcorner))$, where is C is any contradictory statement. Then

$$\not\vdash_T\ Con_T.$$

Proof
Let G the Gödel's sentence of the First Incompleteness Theorem 6.
We prove that

$$\vdash_T\ (Con_T \Leftrightarrow G) \tag{42}$$ G=Con

and use it to prove that $\not\vdash_T\ Con_T$. We conduct the proof by contradiction.
Assume $\vdash_T\ Con_T$. By (42) $\vdash_T\ (Con_T \Leftrightarrow G)$, so $\vdash_T\ G$ what contradicts the First Incompleteness Theorem 6.

To complete the proof we have to to prove now (42). We know by Logic 1 that

$$\vdash_T\ (Con_T \Leftrightarrow G) \text{ if and only if } \vdash_T\ (Con_T \Rightarrow G) \text{ and } \vdash_T\ (G \Rightarrow Con_T).$$

1. We prove the implication $\vdash_T\ (G \Rightarrow Con_T)$.
By definition of Con_T we have to prove now

$$\vdash_T\ (G \Rightarrow \neg Pr_T(\ulcorner C \urcorner)). \tag{43}$$ impl1

The formula C is a contradiction, so $(C \Rightarrow G)$ is a predicate tautology. Hence $\vdash_T\ (C \Rightarrow G)$ and by **D1**

$$\vdash_S\ Pr_T(\ulcorner (C \Rightarrow G) \urcorner).$$

We write **D3** for $A = Pr_T(\ulcorner C \urcorner)$ and $B = \vdash_S \ Pr_T(\ulcorner (C \Rightarrow G) \urcorner)$ and obtain that

$$\vdash_S \ ((Pr_T(\ulcorner C \urcorner) \cap Pr_T(\ulcorner (C \Rightarrow G) \urcorner)) \Rightarrow Pr_T(\ulcorner G \urcorner)). \qquad (44) \quad \boxed{\texttt{2}}$$

We have by Logic 2

$$\vdash_S \ (Pr_T(\ulcorner C \urcorner) \Rightarrow (Pr_T(\ulcorner C \urcorner) \cap Pr_T(\ulcorner (C \Rightarrow G) \urcorner))). \qquad (45) \quad \boxed{\texttt{3}}$$

We get from (45), (44), and Logic 3

$$\vdash_S \ (Pr_T(\ulcorner C \urcorner) \Rightarrow Pr_T(\ulcorner G \urcorner)). \qquad (46) \quad \boxed{\texttt{4}}$$

We apply Logic 4 (contraposition) to the above (46) and get

$$\vdash_S \ (\neg Pr_T(\ulcorner G \urcorner) \Rightarrow \neg Pr_T(\ulcorner C \urcorner)). \qquad (47) \quad \boxed{\texttt{5}}$$

Observe that we by the property (41) in the proof of the First Incompleteness Theorem 3 we have

$$\vdash_S \ (G \Rightarrow \ \neg Pr_T(\ulcorner G \urcorner)). \qquad (48) \quad \boxed{\texttt{6}}$$

We put (47) and (48) together and get

$$\vdash_S \ (G \Rightarrow \ \neg Pr_T(\ulcorner G \urcorner)) \quad \text{and} \quad \vdash_S \ (\neg Pr_T(\ulcorner G \urcorner) \Rightarrow \neg Pr_T(\ulcorner C \urcorner)).$$

Applying Logic 4 to the above we get $\vdash_S \ (G \Rightarrow \neg Pr_T(\ulcorner C \urcorner))$. But by C is by definition Con_T and hence we have proved the $\vdash_S \ (G \Rightarrow Con_T)$ and hence also

$$\vdash_T \ (G \Rightarrow Con_T).$$

2 . We prove now $\vdash_T \ (Con_T \Rightarrow G)$, i.e. the implication

$$\vdash_T \ (\neg Pr_T(\ulcorner C \urcorner) \Rightarrow G). \qquad (49) \quad \boxed{\texttt{impl2}}$$

Here is a concise proof. We leave it to the reader as an exercise to write a detailed version.

By **D2**,

$$\vdash_S \ ((Pr_T(\ulcorner G \urcorner) \Rightarrow Pr_T(\ulcorner Pr_T(\ulcorner G \urcorner) \urcorner))).$$

This implies

$$\vdash_S \ (Pr_T(\ulcorner G \urcorner) \Rightarrow Pr_T(\ulcorner \neg G \urcorner)),$$

by **D1**, **D3**, since $\vdash_S \ (G \Rightarrow \ \neg Pr_T(\ulcorner G \urcorner))$.
This yields

$$\vdash_S \ ((Pr_T(\ulcorner G \urcorner) \Rightarrow Pr_T(\ulcorner (G \cap \neg G) \urcorner)),$$

by **D1**, **D3**, and logic properties, which imples

$$\vdash_S \ ((Pr_T(\ulcorner G \urcorner) \Rightarrow Pr_T(\ulcorner C \urcorner)),$$

by **D1**, **D3**, and logic properties. By Logic 4 (contraposition)

$$\vdash_S \ (\neg Pr_T(\ulcorner G \urcorner) \Rightarrow \neg Pr_T(\ulcorner C \urcorner)),$$

which is $\vdash_S \ (Con_T \Rightarrow G)$ and hence also

$$\vdash_T \ (Con_T \Rightarrow G).$$

This **ends** the proof.

We prove now, as an exercise and reminder, the steps in the proof of part **1.** that follow the predicate logic properties, hence the name Logic. The discovery of needed properties and their proofs for the part **2.** is left as a homework exercise.

Remark 4

By definition 3 the theories T, S are based on a complete proof system for predicate logic and by the monotonicity of classical consequence everything provable there is provable in T, S. In particular all predicate tautologies are provable in T and in S.

L1 **Logic 1**

Given a complete proof system H, for any formulas A, B of the language of H,

$$\vdash (A \Leftrightarrow B) \ \textit{ if and only if } \ \vdash (A \Rightarrow B) \ \textit{ and } \vdash (B \Rightarrow A).$$

Proof
1. We prove implication *if* $\vdash (A \Leftrightarrow B)$, *then* $\vdash (A \Rightarrow B)$ *and* $\vdash (B \Rightarrow A)$.
Directly from provability of a tautology $((A \Leftrightarrow B) \Rightarrow ((A \Rightarrow B) \cap (B \Rightarrow A)))$, assumption $\vdash (A \Leftrightarrow B)$, and MP we get $\vdash ((A \Rightarrow B) \cap (B \Rightarrow A))$. Consequently, from $\vdash ((A \Rightarrow B) \cap (B \Rightarrow A))$, provability of tautologies $((A \cap B) \Rightarrow A)$, $((A \cap B) \Rightarrow B)$ and MP applied twice we get $\vdash (A \Rightarrow B), \vdash (B \Rightarrow A)$.

2. We prove implication *if* $\vdash (A \Rightarrow B)$ *and* $\vdash (B \Rightarrow A)$, *then* $\vdash (A \Leftrightarrow B)$.
Directly from provability of tautology $((A \Rightarrow B) \Rightarrow ((B \Rightarrow A) \Rightarrow (A \Leftrightarrow B)))$, assumption $\vdash (A \Rightarrow B), \vdash (B \Rightarrow A)$, MP applied twice we get $\vdash (A \Leftrightarrow B)$.

L2 **Logic 2** *Given a complete proof system H, for any formulas A, B of the language of H,*

$$\vdash (A \Rightarrow (A \cup B)) \ \textit{ and } \ \vdash (A \Rightarrow (B \cup A)).$$

Proof Directly from predicate tautologies $(A \Rightarrow (A \cup B)), (A \Rightarrow (B \cup A))$ and completeness.

L3 **Logic 3**

Given a complete proof system H, for any formulas A, B of the language of H,

$$\textit{if } \ \vdash (A \Rightarrow B) \ \textit{ and } \vdash (B \Rightarrow C), \ \textit{ then } \ \vdash (A \Rightarrow C).$$

Proof From completeness and predicate respective tautology we get

$$\vdash ((A \Rightarrow B) \Rightarrow ((B \Rightarrow C) \Rightarrow (A \Rightarrow C))). \tag{50}$$ l3

Assume $(A \Rightarrow B)$. Applying MP to (50) twice we get the proof of $(A \Rightarrow C)$.

L4 **Logic 4**

Given a complete proof system H, for any formulas A, B of the language of H,

$$\vdash (A \Rightarrow B) \quad \textit{if and only if} \quad \vdash (\neg B \Rightarrow \neg A).$$

Proof Directly from predicate tautology $((A \Rightarrow B) \Leftrightarrow (\neg B \Rightarrow \neg A))$, completeness and MP.

Guniq **Observation 2**

We proved, a part of proof of the Second Incompleteness Theorem 7 the equivalence (42) which says that the self-referential Gödel sentence G which asserts its own unprovability is equivalent to the sentence asserting consistency. Hence, the sentence G is unique up to provable equivalence (42) and we can say that G is the sentence that asserts its own unprovability.

ω-consistency

We used, in the part (ii) of the First Incompleteness Theorem 6, an additional assumption that $\vdash_T \ Pr_T(\ulcorner G \urcorner)$ implies $\vdash_T \ G$, instead of a habitual assumption of ω-consistency.

The concept of ω-consistency was introduced by Gödel for purpose of stating assumption needed for the proof of his First Incompleteness Theorem 3. The modern researchers proved that the assuption of the ω-consistency can be replaced, as we did, by other more general better suited for new proofs conditions.

Informally, we say that T is ω- **consistent** if the following two conditions are *not* satisfied for any formula A:

(i) $\vdash_T \exists x A(x)$;

(ii) $\vdash_T \neg A(\overline{n})$ for every natural number n.

Formally, ω-consistency can be represented (in varying degrees of generality) by (modification of) the following formula

$$(Pr_T\ (\ulcorner \exists x A(x) \urcorner) \Rightarrow \exists x\ \neg Pr_T(\ulcorner \neg A(x) \urcorner)). \tag{51}$$ omega

3.3 The Formalized Completeness Theorem

Proving completeness of a proof system with respect to a given semantics is the first and most important goal while developing a logic and was the central focus of our study. So we now conclude our book with presentation the formalized completeness theorem 8. We discuss its proof and show how to use it to give new type of proofs, called model-theoretic proofs, of the incompleteness theorems for Peano Arithmetic PA, i.e. for the case when $S = PA$.

Formalizing the proof of completeness theorem for classical predicate logic from chapter **??** within PA we get the following.

Cthm **Theorem 8 (Hilbert-Bernays Completeness Theorem)**

Let U be a theory with a primitive recursive set of axioms.
There is a set Tr_M of formulas such that in $PA + Con_U$ one can prove that this set Tr_M defines a model M of U:

$$\vdash_{PA+Con_U} \forall x(Pr_U(x) \Rightarrow Tr_M(x)) \tag{52}$$ Cmodel

Moreover the set Tr_M is of type Δ_2.

The Hilbert-Bernays Completeness Theorem 8 asserts that modulo Con_U, one can prove in PA the existence of a model of U whose truth definition is of type Δ_2. Its proof is just an arithmetization of the Henkin proof presented in chapter **??**. Following the Henkin proof one adds to the language of U an infinite primitive recursive set of new constants

$$c_0,\ c_1,\ c_2\ \ldots,$$

and adds the axiom (Henkin Axiom)

$$(\exists x A(x) \Rightarrow A(c_{A[x]})) \tag{53}$$ Ha

for each formula $A(x)$. One then enumerates sentences

$$A_0,\ A_1,\ A_2,\ \ldots \tag{54}$$ sent

in this augmented language and defines a *complete theory* by staring with $U$and adding at each step n a sentence A_n, or $\neg A_n$ according to whether A_n is *consistent* with what has been chosen before or not.

The construction is then described within PA. Assuming Con_U one can also prove that the construction never terminates. The resulting set of sentences forms a *complete theory* which by axioms (53) forms a *model* of U. Inspection shows that the truth definition Tr_M of type Δ_2.

The Hilbert-Bernays Completeness Theorem 8 makes possible to conduct new type of proofs of the incompleteness theorems, model- theoretic proofs. Gödel chose as the self-referring sentence a **syntactic statement** " I do not have a proof". He did not want (and saw difficulties with) to use the sentence involving the notion of truth, i.e. the sentence "I am not true". The new proofs use exactly this and this is why they are called **model-theoretic** proofs.

Dana Scott was the first to observe that one can give a **model- theoretic** proof of the First Incompleteness Theorem. Here is the theorem and his short proof.

PA-scott **Theorem 9 (First Incompleteness Theorem)**

Let PA be a Peano Arithmetic.
There is a sentence G of PA, such that

(i) $\nvdash_{PA}\ G$;

(ii) $\nvdash_{PA}\ \neg G$.

Proof
Assume PA is complete. Then, since PA is true, $\vdash_{PA}\ Con_{PA}$ and we can apply the completeness theorem 8 to obtain a formula Tr_M which gives a truth definition for the model of PA. Observe that once PA is complete we have that Pr_{PA} is Tr_M. We choose G by

$$\vdash_{PA} (G \Leftrightarrow \neg Tr_M(\ulcorner G \urcorner)). \tag{55}$$ modG

We claim $\nvdash_{PA}\ G$, $\nvdash_{PA}\ \neg G$. For if $\vdash_{PA}\ G$, then $\vdash_{PA}\ Tr_M(\ulcorner G\urcorner))$. By (**??**) $\vdash_{PA}\ \neg G$. Contradiction. Similarly, $\vdash_{PA} \neg G$ implies $\vdash_{PA}\ G$.

Observe that the sentence G as defined by (55) asserts *"I am not true"*.

Scott 's proof differs from the proof of the First Incompleteness Theorem 6 not only by the choice of the **model- theoretic** method, but also by be a choice of the **model- theoretic** sentence G.

Let's compare these two independent sentences G:
the classic syntactic one of theorem 6 representing statement *" I do not have a proof"* and
the model- theoretic one of theorem 9 representing statement *"I am not true"*.

G-comp **Property 1**

The sentence G_S of the First Incompleteness Theorem 6 asserting its own provability *is*
(i) *unique up to provable equivalence (Observation 2);*
(ii) *the sentence is Π_1 and hence true.*

The sentence G of the First Incompleteness Theorem 6 asserting its own falsity *in the model constructed is*
(iii) *not unique - for the following implication holds*

$$\textit{if } (G \Leftrightarrow \neg Tr_M(\ulcorner G\urcorner)),\ \textit{then } (\neg G \Leftrightarrow \neg Tr_M(\ulcorner \neg G\urcorner)).$$

(iv) *the sentence is* Δ_2 *(theorem 8, and, by* (iii) *there is no obvious way od deciding its truth or falsity.*

Georg Kreisler was the first to present a **model- theoretic** proof of the following.

PA-Kreisel **Theorem 10 (Second Incompleteness Theorem)**

Let PA be a Peano Arithmetic. $\nvdash_{PA}\ Con_{PA}$.

The proof is uses, as did the proof of Hilbert-Bernays Completeness Theorem 8 the arithmetization of Henkin proof of completeness theorem presented in chapter **??**. The proof is carried by contradiction. We assume $\vdash_{PA}\ Con_{PA}$. Then we show, for any presentation of the Henkin proof construction (as given by encoding, the enumeration of sentences (54) ...etc.) there is a number m such that, for any model $\mathcal{N}$ of PA, the sequence of models determined by the given presentations must *stop* after fewer then m steps with a model in which Con_{PA} is false.

4 Homework Problems

1. Follow the proof of Fact 2 for the case of axioms P1 and C1 to prove the case of axioms P2 and C2.
2. Prove the case of axioms P2, C2 and axioms P23, C3 of the Fact 2.
3. Prove Fact 2 in case of axioms P5, C5 and axioms P8, C8 of the Fact 2.
4. Complete the proof of Fact 2 or all cases.
5. We proved that the property $P1'$ of Fact 3 is a generalization of axiom $P1$ of PA (11, i.e. it is provable in PA.

   (i) Write detailed proofs of properties $P2' - P5'$ in PA.

   (i) Write detailed proofs of properties $P6' - P8'$ in PA.
6. Follow the definition 8 and prove the following formulas pre provable in PA for ant terms t, r, s.

   (i) $t \not< t$.

   (ii) $(t < s \Rightarrow (s < r \Rightarrow t < r))$.

   (iii) $(0 < \overline{1}),\ (\overline{1} < \overline{2}),\ (\overline{2} < \overline{3}),\ (\overline{3} < \overline{4}),\ \ldots.$

   (iv) $0 \leq t$.

   (v) $t \leq t$.

   (vi) $(t \leq r \cup r \leq t)$.

   (vii) $(t \leq r \Rightarrow (r \leq t \Rightarrow t = r))$.

7. Follow the definition 8 and prove the following formulas pre provable in PA for ant terms t, r, s.

   (i) $(t \leq s \Rightarrow (s \leq r \Rightarrow t \leq r))$,

   (ii) $(t \leq s \Rightarrow (t + r \leq s + r))$,

   (ii) $(r > 0 \Rightarrow (t > 0 \Rightarrow r \cdot t > 0))$.

8. Let RR be the Robinson System (23). Show that RR is a proper subtheory of PA by finding a model of RR that is not a model for PA.

9. Let RR be the Robinson System (23). Let n, m be any natural numbers. Prove the following holds in RR.

   (i) If $m \neq n$, then $\overline{m} \neq \overline{n}$.

   (ii) $\overline{m+n} = \overline{m} + \overline{n}$ and $\overline{m \cdot mn} = \overline{m} \cdot \overline{n}$ are provable in RR .

   (ii) Any model for RR is infinite.

10. Here us the reasoning we used explaining Hilbert Consistency Program.

    " Let A be a *real* statement $\forall x(f(x) = g(x))$. Assume $\vdash_I A$. Then there is a derivation d of A in I. But, derivations are concrete objects and, for some real formula $P(x, y)$ *encoding* derivations in I, $\vdash_R P(d, \ulcorner A \urcorner)$, where $\ulcorner A \urcorner$ is some *code* for A.
    Now, if A were false, one would have $f(a) \neq g(a)$ for some a and hence $\vdash_R P(c, \ulcorner \neg A \urcorner)$ for some c. In fact, one would have a stronger assertion $\vdash_R (f(x) \neq g(x) \Rightarrow P(c_x, \ulcorner \neg A \urcorner))$. But, if R proves *consistency* of I, we have

    $$\vdash_R \neg(P(d, \ulcorner A \urcorner) \cap P(c, \ulcorner \neg A \urcorner)),$$

    whence $\vdash_R f(x) = g(x)$, with free variable x, i.e. $\vdash_R \forall x(f(x) = g(x))$."

    (i) Write down a detailed proof of correctness of the last part of reasoning:

    "But, if R proves *consistency* of I, we have

    $$\vdash_R \neg(P(d, \ulcorner A \urcorner) \cap P(c, \ulcorner \neg A \urcorner)),$$

    whence $\vdash_R f(x) = g(x)$, with free variable x, i.e. $\vdash_R \forall x(f(x) = g(x))$."

    (ii) List, prove and use proper Logic Properties similar to properties Logic 1 - Logic 4 in the proof of Theorem 7.