Predicate Logic
PART 2

CSE 352 Artificial Intelligence

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Lecture Notes

Predicate Logic Part 2

- PREDICATE LOGIC TAUTOLOGIES
- BASIC LAWS OF QUANTIFIERS

Basic Laws of Quantifiers (Predicate Logic Tautologies)

De Morgan Law

$\neg \forall x\, A(x) \equiv \exists x\, \neg A(x)$

$\neg \exists x\, A(x) \equiv \forall x\, \neg A(x)$

where A(x) is any formula with free variable x

≡ means “logically equivalently”

Definability:

$\neg \forall x\, A(x) \equiv \exists x\, \neg A(x)$

$\neg \exists x\, A(x) \equiv \forall x\, \neg A(x)$

Application Example: A(x) is $((Px) \wedge \neg R(x)) \neg Q(x,y))$

$\equiv \exists x(P(x) \wedge \neg R(X) \wedge \neg Q(x,y))$

Example (Mathematical Formula)

Laws Application:

$\neg \forall x((x>0 \Rightarrow x+y > 0) \wedge \exists y\,(y>0))$

$\equiv$ (by De Morgan's Law)

$\exists x((x>0 \wedge x+y > 0) \wedge \exists y\,(y > 0))$

$\equiv \exists x((x>0 \wedge x+y \leq 0) \vee \forall y(y \geq 0))$

$\neg\,(A \Rightarrow B) \equiv (A \wedge \neg B)$, $\neg(A \wedge B) \equiv (\neg A \vee \neg B)$

$\neg(x + y) > 0) \equiv x+y \leq 0$

$\neg\,\exists y\,(y >< 0) \equiv \forall y\,\neg\,(y < 0)$

$\equiv \exists y\,(y \geq 0)$

Logic Formula (corresponding to the Math formula

$\neg\,\forall x(A(x) \Rightarrow B(\,x,y)) \wedge \exists y\,C(y))$

$\equiv \exists x\,\neg((A(x) \Rightarrow B(x,y) \wedge \exists y\,C(y))$

$\equiv \exists x((A(x) \wedge \neg B(\;x,y)\,) \vee \neg\,\exists y\,C(y))$

$\equiv \exists x\,((A(x) \wedge \neg B(\,x,y)) \vee \forall y\,\neg C(y))$

Distributivity Laws (to be proved)

1. $\exists x(A(x) \vee B(x)) \equiv (\exists x\, A(x) \vee B(x))$
   Existential quantifier is distributive over $\vee$ $(\exists x, \vee)$
2. $\forall x\,(A(x) \wedge B(x)) \equiv (\forall x\, A(x) \wedge B(x))$ universal quantifier is distributive over $\wedge$ $(\forall x\,, \wedge)$
3. Existential quantifier is distributive over $\wedge$ in only one direction

$\exists x(A(x) \wedge B(x)\,) \Rightarrow (\exists x\, A(x) \wedge \exists x\, B(x))$

It is not true, that for any $x \neq \phi$ and any A(x), B(x) $(\exists x\, A(x) \wedge \exists x\, B(x)) \Rightarrow \exists x(A(x) \wedge B(x)\,)$

Example: $x \in R$ for x = R A(x) > 0, B(x) = x^2

$\exists x\,(x>0) \wedge \exists x(x>0)$ is a **true** statement!

$\exists x(x>0 \wedge x< 0)$ is **false**!

Distributivity (continued)

4. Universal quantifier is distributive over ∧ in only one direction:

$((\forall x\ A(x) \lor \forall x\ B(x)) \Rightarrow \forall x(A(x) \lor B(x)))$

$A(x) = x < 0 \quad B(x) = x \geq 0$

Interpretation T ⇒ F = F

Example: $x \in R$ for $x = R$

$\forall x\ (x>0 \lor x \geq 0)$ is **true**

$\forall x(x<0) \lor \forall x(x \geq 0)$ **false**

5. Universal quantifier is distributive over ⇒ in one direction:

$\forall x(A(x) \Rightarrow B(x)) \Rightarrow (\forall x\ A(x) \Rightarrow \forall x\ B(x))$

Example: Take $x \in R$

$(\forall x(x < 0) \Rightarrow \forall x(x+1 > 0)$ is False

Take x= -2, we get $(-2 < 0 \Rightarrow -2+1 > 0)$ False

Introduction and Elimination Laws

B- Formula without free variables

6. $\forall x(A(x) \Rightarrow B) \equiv (\exists x\, A(x) \Rightarrow B)$

7. $\exists x(A(x) \Rightarrow B) \equiv (\forall x\, A(x) \Rightarrow B)$

8. $\forall x(B \Rightarrow A(x)) \equiv (B \Rightarrow \forall x\, A(x))$

9. $\exists x(B \Rightarrow A(x)) \equiv (B \Rightarrow \exists x\, A(x))$

10. $\forall x(A(x) \vee B) \equiv (\forall x\, A(x) \vee B)$

11. $\forall x(A(x) \wedge B) \equiv (\forall x\, A(x) \wedge B)$

12. $\exists x(A(x) \vee B) \equiv (\exists x\, A(x) \vee B)$

13. $\exists x(A(x) \wedge B) \equiv (\exists x\, A(x) \wedge B)$

Remark: we prove 6 -9 from 10 – 13 + de Morgan + definability of implication

Intuitive (not very formal) Semantics for Predicate Logic

We can use truth sets for predicates $x \neq \phi$

$\{x \in X: P(x)\}$ is called a truth set for the predicate P(x).

Example1:

P(x): x+1 = **3 interpretation of P(x)** in x x={3, 4}

x={1, 2, 3}

$\{x \in X: P(x)\} = 2$, $\{x \in X: P(x)\} = \phi$

Example2:

P(x): $x^2 \leq 0$ **Interpretation of P(x)**

x = N x = $P^{+}-\{0\}$

$\{x: P(x) = \{0\}$ $\{x:P(x)\} = \phi$

Intuitive (not very formal) semantics for Predicate Logic

We use truth sets for predicates x ≠ ϕ

Conjunction:

{x∈ X: (P(x) ∧ Q(x))} = {x: P(x) ∧ {x: Q(x)}

∧ is used to symbolize conjunction, known as an intersection

Disjunction:

{x∈ X: (P(x) ∨ Q(x))} = {x: P(x) ∨ {x: Q(x)}

∨ is used for disjunction, known as union

Negation:

{x∈ X: ¬P(x)} = X - {x∈ X: P(x)}

¬ is the negation and – is the complement

Intuitive (not very formal) semantics for Predicate Logic

Implication:

$\{x \in X: (P(x) \Rightarrow Q(x))\} \equiv X - \{x:P(x)\} \vee \{x : Q(x)\}$

$= - x:P(x)\} \vee \{x : Q(x)\}$

$=\{x: \neg P(x)\} \vee \{x : Q(x)\}$ Interpretation

Example:

$\{x \in N\ n>0 \Rightarrow n^2 <0\} = \{x \in N\ x \leq 0\} \vee \{\{x \in N : n^2<0\}$

$=\{0\} \vee \phi = \{0\}$

Truth Sets for Quantifiers

Definition:

$\forall x\, A(x) = T$ iff $\{x \in X: A(x)\} = X\}$ in the domain $X \neq \phi$ where $A(x)$ is any formula with x-free

Definition:

$\forall x\, A(x) = F \quad (X \neq \phi)$

iff $\{x \in X: A(x)\} \neq X$

where $A(x)$ is any formula with x-free variable

Truth Sets for Quantifiers

Definition:

$\exists x\, A(x) = T$ (in $x \neq \phi$) iff $\{x \in X : A(x)\} \neq \phi$

Definition:

$\exists x\, A(x) = F$ (in $x \neq \phi$) iff $\{x \in X : A(x)\} = \phi$

A(x) is a formula (complex) with free variable x.

Remark

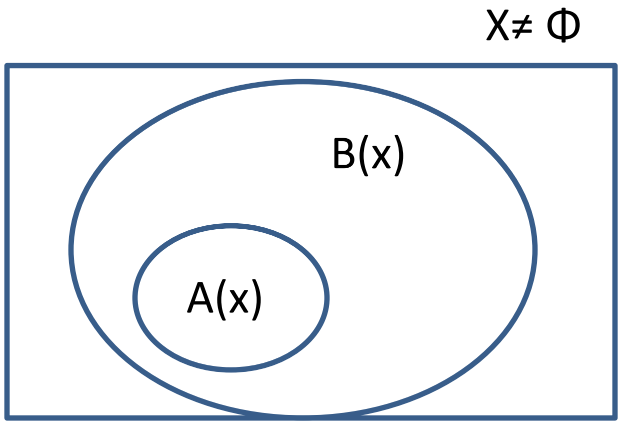
Observe that

$\forall x\ (A(x) \Rightarrow B(x)) = T \quad X \neq \Phi$

Iff $\{x \in X : A(x) \Rightarrow B(x)\} = X$

Iff $\{x : A(x)\} \subseteq \{x : B(x)\}$

Picture



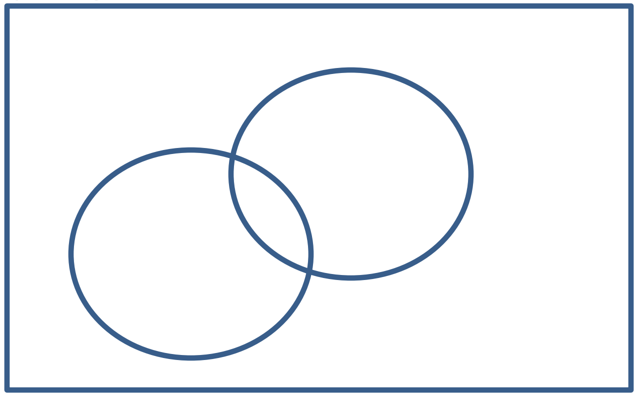
Venn Diagrams For Quantifiers

Venn Diagrams For Quantifiers

$\exists x(A(x) \wedge B(x))=T$ iff $X \neq \Phi$

$\{x:A(X)\} \wedge \{x:B(x)\} \neq \Phi$

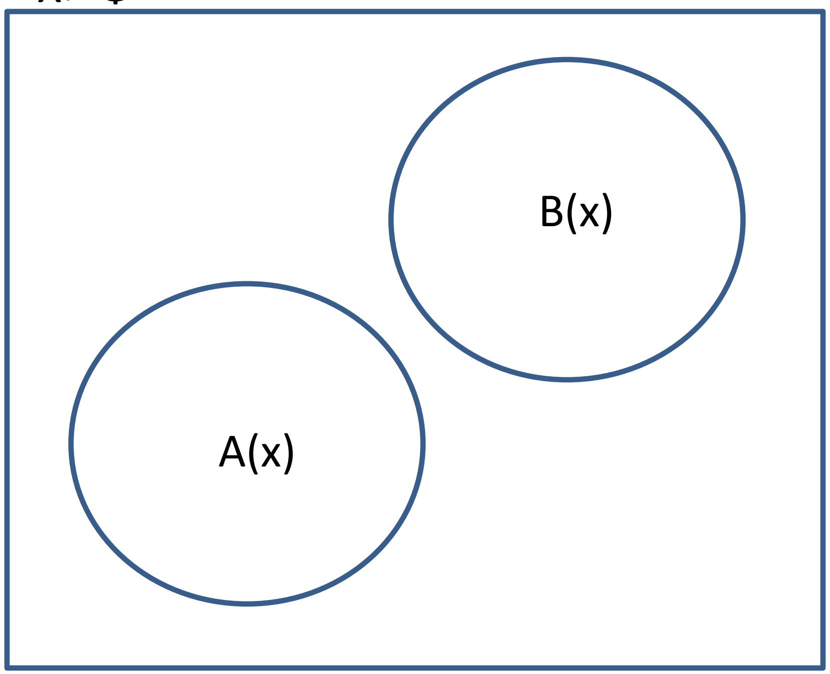
Picture



$\exists x(A(x) \wedge B(x)) = F$ iff $\{x:A(x) \wedge \{x:B(x)\} = \Phi$

Pictures

X≠ Φ



Remember {x:A(x)}, {x:b(x)}
Can be Φ!

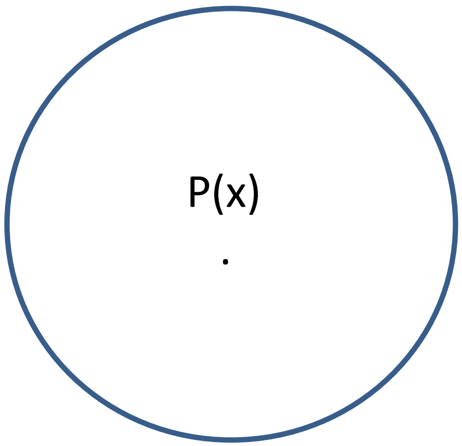
Example:

Draw a picture for a situation where (in $X \neq \Phi$)

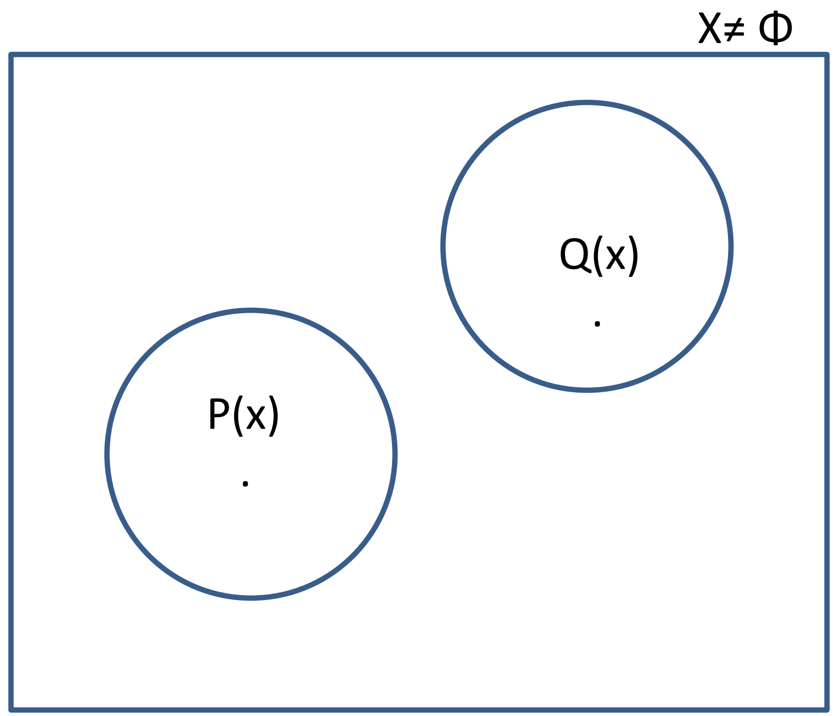
1. **$\exists x\, P(x) = T$,**
2. **$\exists x\, Q(x) = T$,**
3. **$\exists x(P(x) \wedge Q(x)) = F$ and**
4. **$\forall x\, (P(x) \vee Q(x) = F$**

1. $\exists x\, P(x) = T$ iff $\{ x{:}P(x)\} \neq \Phi$
2. $\exists x\, Q(x) = T$ iff $\{x{:}Q\,(x)\} \neq \Phi$
3. $\{x{:}P(x)\} \wedge \{x{:}Q(x)\} \neq \Phi$
4. $\{x{:}P(x)\} \vee \{ x{:}Q(x)\} \neq X$

Picture:



Denotes P(x) ≠ Φ



Proving Predicate Tautologies

Prove that

$\vdash(\forall x\ A(x) \Rightarrow \exists x\ A(x))$

Proof:

Assume that not True

(Proof by contradiction) i.e. that there are $X \neq \Phi, A(x)$ such that.

$(\forall x\ A(x) \Rightarrow \exists x\ A(x))$ **= is F**

iff **$\forall x\ A(x)=T$** and **$\exists x\ A(x)=F$** $(A \Rightarrow B) = F$

iff (def) $x \neq \phi$ iff (def)

$\{x \in X : A(x)\} = X$ and **$\{x \in X : A(x)\} = \phi$**

iff $x = \phi$

Both of these formulas are a contradiction with **$x \neq \phi$** Hence proved!!

Prove:

$\neg \forall x\, A(x) \equiv \exists x\, \neg A(x)$

$\exists x\, \neg A(x) = T$ in $X \neq \phi$ **iff** $\{x: \neg A(x)\} \neq \phi$ when $B = \phi$ then $B \neq x$ as $x \neq \phi$ $B \neq \phi$

$X - \{x: A(x)\} \neq \phi$

iff $X - B \neq \phi$

$\{x: A(x)\} \neq x$ **iff**

iff $B \neq x$

$\forall x\, A(x) = F$ we assume that for any A(x)

iff $\{x \in X: A(x)\}$ exists

$\neg \forall x\, A(x) = T$

Example: B – No variable x

$\forall x\ (A(x) \vee B) \equiv \forall x\ A(X) \vee B$

= T **iff**

{x: A(x) ∨ {x: B} = X

iff {x: A(x) = X

or B=X

it means

$\forall x\ \ A(x)$=T or B=T

and

$\forall x\ (A(x) \vee B)$ = T

Prove

$\exists x(A(x) \land B(x)) \equiv \exists x\, A(x) \land \exists x\, B(x)$

$\exists x(A(x) \land B(x)\} = T$ iff

$\{x: (A(x) \land B(x)\} \neq \phi$ (definition)

$= \{x: (A(x)\} \land \{x: (B(x)\} \neq \phi$ iff $A \land B \neq \phi$

iff $A \neq \phi$ and $B \neq \phi$

$\{x: A(x)\} \neq \phi$ and $\{x: B(x)\} \neq \phi$

$= \exists x\, A(x)=T$ and $\exists x\, B(x)=T$