Propositional Resolution Introduction

(Nilsson Book Handout)

Professor Anita Wasilewska
CSE 352 Artificial Intelligence

Propositional Resolution
Part 1

SYNTAX "dictionary"

Literal – any **propositional** VARIABLE a or negation of a variable ¬ a, for a ∈ VAR

Example: variables: a, b, c .... negation of variables: ¬a, ¬b, -d ...

Positive Literal: any variable a ∈ VAR

Clause – any **finite set** of **literals**

Example: C1, C2, C3 are clauses where

C1 = {a, b} , C2 = {a, ¬ c} , C3 = { a, ¬a, ..., a_k }

Syntax "Dictionary"

Empty Clause: {} is an empty set i.e. a clause without elements

Finite set of clauses

CL = { C1, …., Cn}

Example

CL = {{a}, { }, { b, ¬a} , {c, ¬ d}}

Semantics – Interpretation of Clauses

- Think **semantically** of a clause
- $C = \{ a_1, \ldots\ldots, a_n\}$ as disjunction, i.e.

  C is logically equivalent to

  $a_1 \cup a_2 \cup \ldots\ldots \cup a_n$ $a_i \in$ Literal
- **Formally** – given a truth assignment $v : VAR \rightarrow \{0, 1\}$ we extended it to set of all CLAUSES **CL** as follows:

$$v^* : \mathbf{CL} \rightarrow \{0, 1\}$$

$$v^*(C) = v^*(a_1) \cup \ldots\ldots \cup v^*(a_n)$$

for any clause C in **CL**, where

0 – False, 1 – True

Shorthand : $v^* = v$

Satisfability, Model, Tautology

Example: let v : VAR -> {0, 1} be such that

- v(a) = 1, v(b) = 1 , v(c)= 0 and let

C = { a, ¬ b, c, ¬a}

We evaluate :

v(C) = v(a) U ¬v(b) U v(c) U ¬v(a) =

1 U 0 U 0 U 1 = 1

OBSERVE that v(C) =1 for all v, i.e. the clause

C = { a, ¬ b, c, ¬a} is a **Tautology**

Satisfability, Model, Tautology

Definitions

1. For any clause **C**, and any truth assignment **v** we write **v l= C** and say that **v** **satisfies** **C** iff **v(C) =1**

2. Any v such that **v l= C** **is called** **a MODE L** **for C**

3. A clause C is **satisfiable** iff it has a **MODEL**, i.e.

C is satisfiable iff there is a v such that v l= C

4. A clause C is a **tautology** iff **v l= C** for all v,

i.e all truth assignments **v** are **models for C**

Notations

- a, a, a is a finite sequence of 3 elements
- {a, a, a} = {a} is a finite set
- a, b, c ≠ b, a, c are different sequences
- {a, b, c} = {b, a, c} are the same sets
- {a, a, b, c} is a multi – set (if needed)

Sets of Clauses CL

DEFINITIONS

1. A clause C is unsatisfiable iff it has no MODEL

i.e. v(C) =0 for all truth assignments v

Remark: **the empty clause {} is the only unsatisfiable clause**

Let **CL** = { C_1, ...,C_n) be a **finite set of clauses**.

2. We extended v : VAR -> {0, 1} to any set of clauses CL

$$v (\mathbf{CL}) = v(C_1) \wedge .... \wedge V(C_n)$$

A finite set of clauses CL is semantically equivalent to a conjunction of all clauses in the set CL

Unsatisfiability

Definitions

1. A set of clauses **CL** is **satisfiable** iff it **has a model**, i.e. iff $\exists v \; v(\mathbf{CL}) = 1$

2. A set of clauses **CL** is **unsatisfiable** iff it **does not have a model**, i.e. iff $\forall v \; v(\mathbf{CL}) = 0$.

Remark:

If $\{\} \in CL$ then CL is unsatisfiable

Unsatisfability

Consider a set of clauses

CL = {{a}, {a,b}, {¬ b}}

CL is **satisfiable** because any v, such that

v(a) =1, v(b) =0 is a **model** for **CL**

Check: v(CL) = 1 ∧ (1 U 0) ∧ 1 = 1

FACT: When {a} and {¬ a} are in **CL,**

then the set **CL** is **unstisfiable**

Remember: (a ∧ ¬ a) is a contradiction

Syntax and Semantics

- Example:
- C1 = { a, b, ¬c}, C2 ={c , a} - syntax
- C1 = a U b U ¬c - semantics
- C2 = c U a - semantics

- **CL** = {C1, C2} = {{a , b, ¬c} , {c , a}} – syntax

CL = (a U b U ¬c) ∧ (c U a) - semantics

Syntax and Semantics

Definitions:

CL is satisfiable iff there is v, such that v(CL) = 1

CL is unsatisfiable iff for all v, v(CL) = 0

- **CL** = { C1,C2,…… ,Cn} - synatx
- **CL** = C1 ∧ …… ∧ Cn - semantics

Semantical Decidability

- A statement:
- " A finite set **CL** of clauses is/ is not satisfiable"

  is a **decidable statement**.
- **CL** has n propositional variables, hence we have 2^n possible truth assignments v to examine and evaluate whether **v(CL) = 1 or v(CL) = 0**
- This is called **Semantical Decidability**
- **Problem:** Exponential complexity

Syntactical Decidability Method: Resolution Deduction

- **Goal** : We want to show that a finite set **CL** of clauses is **unsatisfiable**
- **Method : Resolution deduction** :
- **Start** with **CL;** apply a transformation rule called **Resolution** as long as it is possible.
- **If** you **get {},** then answer is **Yes**, i.e.

  CL is **unsatisfiable**
- **If** you **never get {}** , then answer is **NO**, i.e **CL** is **satisfiable**

Resolution **Completeness Theorem 1**

Completeness of the Resolution:

CL is **unsatisfiable** **iff** we obtain the empty clause **{}** by a multiple use of the **Resolution Rule**

- **Symbolically:** **CL ⊢ {}**
- It means we **deduce** the empty clause **{}** from **CL** by use of the resolution rule;
- We **prove** **{}** from **CL** by resolution

Resolution **Completeness Theorem 1**

|= CL denotes CL is a tautology

=| CL denotes CL is unsatisfiable (contradiction)

- We write symbolically:

Resolution Completeness Theorem 1

=| CL iff CL $\vdash$ {}

Refutation

- **Refutation: proving the contradiction**

In classical logic we have that:

A **formula A is a tautology iff ¬A is a contradiction**

Symbolically: **$\models A$ iff $=\!\mid \neg A$**

Observe:

$\models (A1 \wedge \ldots\ldots \wedge An \Rightarrow B)$ iff $=\!\mid (A1 \wedge \ldots\ldots \wedge An \wedge \neg B)$

Because **$\neg (A \Rightarrow B) \equiv (A \wedge \neg B)$**

Refutation

By **Resolution Completeness Theorem** this is **almost** equivalent to

|= (A1 ∧ …… ∧ An => B) iff (A1 ∧ …… ∧ An ∧ ¬B) ⊢ {}

Almost- means not YET Resolution works for **clauses** not formulas!

The **IDEA** is the following:

to prove B from A1, ….,An we keep A1,….., An , ADD ¬B to it and use the Resolution Rule

If we get {}, we have proved (A1 ∧ …… ∧ An => B)

It is called a proof by REFUTATION; to prove C we start with ¬C and if we get a contradiction {}, we have proved C

Formulas – Clauses

Resolution works only for clauses

To use **Resolution Deduction** we need to **transform** our **formulas** into **clauses** i.e. we need to **prove** the following

Theorem

For any formula $A \in F$, there is a **set of clauses** CL_A such that A is logically equivalent to the set of clauses CL_A

CL_A is called a **clausal form** of the formula A

We have good **set of rules** for automatic transformation of A into its **clausal form** and we will study it as next step

Completeness

- **Resolution Completeness 2**

  For any propositional formula A

  $\models A$ **iff** $CL_{\neg A} \vdash \{\}$

  where $CL_{\neg A}$ is the clausal form of $\neg A$

- **Resolution Proof of A definition:**

  $\vdash_R A$ **iff** $CL_{\neg A} \vdash \{\}$

Resolution Completeness 2:

$\models A$ **iff** $\vdash_R A$

Resolution Rule R

- $C_1(a)$ means: clause C_1 contains a positive literal **a**
- $C_2(\neg a)$ means: clause C_2 contains a negative literal **¬a**

- **Resolution Rule R** (two Premises)

  $$\frac{C_1(a) \; : \; C_2(\neg a)}{(C_1 - \{a\} \cup C_2 - \{\neg a\})}$$

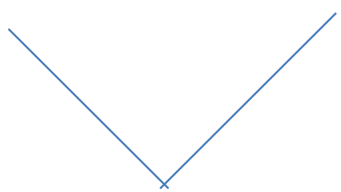
  Resolve on a

  $(C_1-\{a\} \; U \; C_2-\{\neg a\})$ <- **Resolvent**

Clauses $C_1(a)$ and $C_2(\neg a)$ are called a **complementary pair**

Resolution Rule

- **Resolution Rule** takes **2 clauses** and returns **one**. We usually write it in a form of a **graph**:
- **Definition:** $C_1(a)$, $C_1(\neg a)$ is called a complementary pair
- $C_1(a)$ $C_1(\neg a)$



Resolve on a

$(C_1\text{-}\{a\})$ U $(C_2\text{-}\{\neg a\})$ **<-** Resolvent on a

Resolution Rule R

- Clauses are SETS!
- $\{C_1, C_2\}$ Complementary Pair

$C_1 = \{a, b, c, \neg d\}$ $C_2 = \{\neg a, \neg b, d\}$

Resolve

on a

$\{b, c, \neg d, \neg b, d\}$ Resolvent on a

Example

$C_1 = \{a, b, c, \neg d\}$ $C_2 = \{\neg a, \neg b, d\}$

$\{a, c, \neg d, \neg a \quad d\}$

- **Resolution Rule: R (Two Premises)**

$\underline{C_1(b) : C_2(\neg b)}$ **Resolve on b**

$(C_1 - \{b\} \cup C_2 - \{\neg b\})$ <- Resolvent

Exercise

- **CL** - set of clauses

Find all resolvents of **CL**

It means locate all clauses in **CL** that are Complementary Pairs and Resolve them

$C_1 = \{a, b, c, \neg d\}$ $C_2 = \{\neg a, \neg b, d\}$

CL $= \{C_1, C_2\}$ has 3 Complementary Pairs

$C_1(a), C_2(\neg a)$ – P1

$C_1(b), C_2(\neg b)$ – P2

$C_2(d), C_1(\neg d)$ – P3

Example

- **CL** $=\{C_1, C_2\} = \{C_2, C_1\}$

$C_1=\{a, b, c, \neg d\}$ $C_2=\{\neg a, \neg b, d\}$

Remember:

Resolution Rule uses **one literal** at the time!

$C_1(a)$; $C_2(\neg a)$ **Resolve on a :** we get $\{b, c, \neg d, \neg b, d\}$

$C_1(b)$; $C_2(\neg b)$ **Resolve on b :** we get $\{a, c, \neg d, \neg a, d\}$

$C_1(d)$; $C_2(\neg d)$ **Resolve on d :** we get $\{a, b, c, \neg a, \neg b\}$

Example

<u>$C_1(b) : C_2(\neg b)$</u> Pair $\{C_1\ C_2\}$

$(C_1\text{-}\{b\})\ U\ (C_2\text{-}\{\neg b\})$

{a, b, c, ¬ d} {¬ a ,¬b ,d}

Resolve on b

{a, c, ¬d, ¬a, d} <- Resolvent on b

Example

$\underline{C_1(d) : C_2(\neg d)}$ on $\{C_1\ C_2\}$

$(C_1-\{d\})\ U\ (C_2-\{\neg d\})$

{a, b, c, ¬ d} ;{¬ a ,¬b ,d}

Resolve on d

{a, b , c, ¬a, ¬ b}

Example

$C_1 = \{a, b, c, \neg d\}$; $C_2 = \{\neg a, \neg b, c, d\}$

Resolve on b

$\{a, c, \neg d, \neg a, d\}$

Two clauses (one complementary pair) **can have more than one resolvent** – you can also resolve the complementary pair C_1 C_2 on a

Example

- We can also **resolve** $\{C_1, C_2\}$ on a

{a, b, c, ¬ d} , {¬ a ,¬b ,d} $\{C_1, C_2\}$

Resolve on a

{b , c, ¬d, ¬ b, d}

These are all resolvent of pair $\{C_1\ C_2\}$:

{b , c, ¬d, ¬ b, d}, { a, c, ¬d, ¬ a, d}

{a, b, c, ¬ a, ¬ b}

Resolution Deduction

- **CL** - set of clauses

Procedure: Deduce a clause C from **CL:** **CL** ⊢$_R$ {C}

Start with **CL**, apply the resolution rule R to **CL**

Add resolvent to **CL** and

Repeat adding resolvents to already obtained set o fresolvents

until you get C

Example

CL = {{ a, b}, { ¬ a, c}, { ¬ b, c}}

R on a {b ,c}

R on b

{c }

CL ⊢$_R$ {c }

Example

- **CL** = {{ a, b},{ ¬ a, c}, { ¬ b, c}}

Resolve on b

{a, c}

Resolve on a

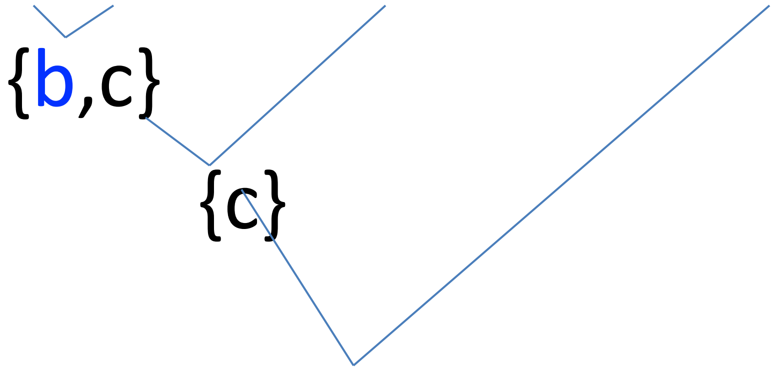
{c}

We have 2 possible **deduction** of { c } from **CL**

CL $\vdash_R$ { c }

Example

- **CL** = {{ a, b},{ ¬ a, c},{ ¬ b, c}, {¬ c}}



{} **CL** $\vdash_R$ {}

CL is unsatisfiable by **Completeness Theorem**

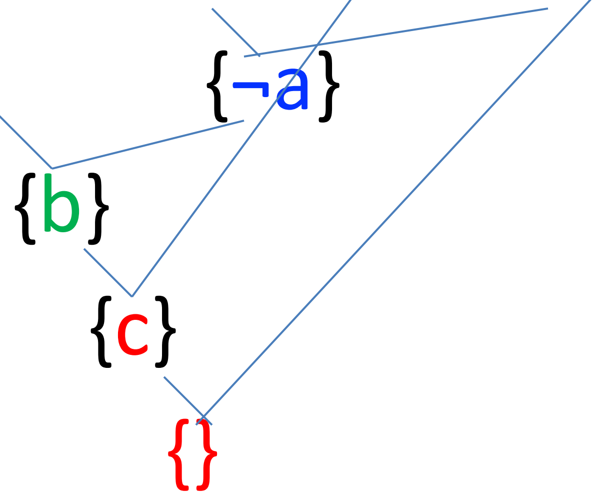
$=|$ **CL** **iff** **CL** $\vdash_R$ {}

Resolution deduction is not unique!

Next: Strategies for Resolution

Example

- **CL**= {{ a, b},{ ¬ a, c},{ ¬ b, c}, {¬ c}}



Another deduction of {} from **CL**

Exercise

- Let **CL** = {{ a, b},{ ¬ a, c},{ ¬ b, c}}

Find all possible deduction from **CL**

Remember:

1. If you get {}, it means **CL** is unsatisfiable.
2. If you **never** get {}, it means **CL** is satisfiable.

1 and 2 is true by **Completeness Theorem:**

$\models$ **CL** iff **CL** $\vdash$ {}

CL is unsatisfiable **iff** there is a deduction of {} from **CL**

CL is satisfiable **iff** there is NO deduction of {} from **CL**

Exercise

- **CL** = {{ a, b},{ ¬ a, c},{ ¬ b, c}}

Derivation 1: {{ a, b},{ ¬ a, c},{ ¬ b, c}}

R on a {b, c}

{c} R on b **STOP**

Derivation 2: {{ a, b},{ ¬ a, c},{ ¬ b, c}}

R on b {a, c}

{c} R on a **STOP**

No more (possible) Derivations, i.e. by Completeness Theorem we have that

CL is satisfiable

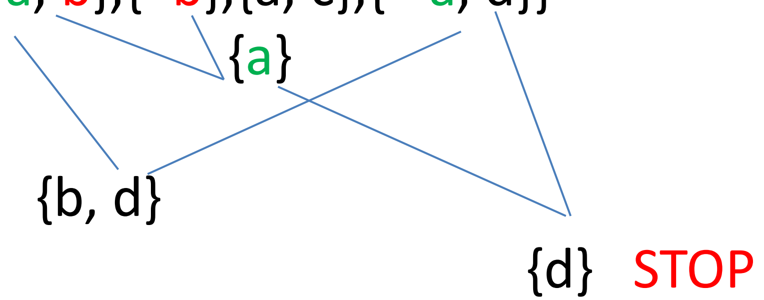
Exercise

- **CL** is **unsatisfiable** iff there is deduction of {} from it, i.e.

  CL $\vdash_R$ {}

CL is **satisfifable** iff never **CL** $\vdash_R$ {} (must cover all possibilities of deduction)

CL = {{ a, b},{¬b},{a, c},{ ¬a, d}}



This is just **one** derivation.
You must consider ALL possible derivations and show that none ends with {} to prove that CL is **satisfiable**

Exercise

- **Given:** **CL** = $\{C_1, C_2, C_3, C_4\}$

CL =$\{\{a, b, \neg b\}, \{\neg a, \neg b, d\}, \{a, b, \neg c\}, \{\neg a, c, b, e\}\}$

1. Find all complementary pairs . Here they are:

$\{C_1, C_2\}$ $\{C_1, C_4\}$,

$\{C_3, C_2\}$ $\{C_2, C_3\}$,

$\{C_3, C_4\}$, $\{C_2, C_4\}$

2**. Find all resolvents** for your **complementary pairs**

For example: $C_1 = \{a, b, \neg b\}$, $C_2 = \{\neg a, \neg b, d\}$ has 2 resolvents.

Resolve on a: $\{\neg b, d, b\}$

Resolve on b;

$\{a, \neg a, d, \neg b\}$

Exercise

- **CL** = {C_1, C_2}, for C_1 ={a ,b ,c ,¬d}, C_2 = {¬ a ,¬ b, d}

  CL has 3 resolvents :-

  1. {¬a ,¬b , a, b, c} – resolve on d
  2. {¬a ,c ,¬d , d, a} – resolve on b
  3. {b, c ,¬d ,d} – resolve on a

Let now **CL** = {C_1, C_2, C_3}, for C_1={a}, C_2={b, ¬a}, C_3={¬b, ¬a}

Exercise:

Find all Complementary Pairs + find all their resolvents

Propositional Resolution
Part 2

GOAL: Use Resolution to prove/ disapprove $\models A$

PROCEDURE

Step 1: Write $\neg A$ and transform $\neg A$ info set of clauses $\mathbf{CL}_{\{\neg A\}}$ using Transformation rules

Step 2: Consider $\mathbf{CL}_{\{\neg A\}}$ and look at if you can get a deduction of $\{\}$ from $\mathbf{CL}_{\{\neg A\}}$

ANSWER

1. $\mathbf{CL}_{\{\neg A\}} \vdash_R \{\}$ – Yes, $\models A$
2. $\mathbf{CL}_{\{\neg A\}} \nvdash \{\}$ (i.e. you never get $\{\}$) – No, not $\models A$

Rules of transformation

- **Rules of transformation** of a formula A into a logically equivalent set of clauses CL_A
- **Rule (U): (AUB) + Information**

What "Information" mean?

Example: a, b, (a U ¬(a=> b)), ¬c

a, b, a ,¬(a=> b), ¬c

a,b and ¬c is Information

Rule (U) : I , (AUB), J

I, A, B, J

I,J --- Information around

Implication Rule (=>)

- **I, (A=>B), J** **(A=>B)**

I, ¬A, B, J **¬A , B**

Example: a, (a U b), (a => ¬a), (a ∧ b), c

(=>)

a, (a U b), ¬ a, ¬ a, (a ∧ b), c

(U)

a, a, b, ¬ a, ¬ a, (a ∧ b), c

next step?

we need (∧) Rule!

Conjunction Rule (∧)

I, (A ∧B), J

(∧)

I , A, J   I, B, J

(A∧B)

(∧)

A   B

Example:

a, a, b, ¬a, ¬a, (a ∧ b), c

(∧)

a, a, b, ¬a, ¬a, a, c   a, a, b, ¬a, ¬a, b, c

STOP when get only literals – called leaves

Form clauses out of the leaves

Set of Clauses

Procedure: Leaves – to – Clauses

1. make SETS out of each leaf;

each leaf becomes a **clause C**

2. make a set of clauses **CL** as a **set of all clauses** C obtained in 1.

Leaf 1: {a, a, b, ¬a, ¬a, a, c} = {a, b, ¬a, c}

Leaf 2: {a, a, b, ¬a, ¬a, b, c} = {a, b, ¬a, c}

- Observe that we end-up with only **one set** of clauses
- **CL** ={Leaf 1, Leaf 2} = { {a, b, ¬a, c} }

Negation of Implication Rule (¬ =>)

I, ¬ (A =>B), **J**   ¬ (A => B)

(¬=>)

I , A, **J**   **I** , ¬B, **J**   A   ¬B

Example:

a, b, a, ¬ (a => b), ¬ c

a, b, a, a, ¬c   a, b, a, ¬b, ¬c

Stop – when only literals :

Form clauses out of leaves a, b, a, a, ¬c and a, b, a, ¬b, ¬c

Clauses

- Leaf1: a, b, a, a, ¬c makes clause {a, b, ¬c}
- Leaf 2: a, b, a, ¬b, ¬c makes clause {a, b, ¬b, c}

- **CL** = {{a, b, ¬c}, {a, b, ¬b, c}}

- **CL** is set of clauses corresponding to

  a, b, a, ¬ (a => b), ¬ c

Negation of Disjunction Rule (¬ U)

I, ¬(A UB), J

(¬U)

I , ¬A, J   I , ¬B, J

¬(A U B)

¬A   ¬B

- **Rule (¬ U)** **c**oresponds to DeMorgan Law:

  ¬(AUB) ≡ (¬A ∧ ¬B)

Negation of Conjunction Rule (¬∧)

I, ¬(A ∧B), J

| (¬∧)

I , ¬A, ¬B, J

¬(A ∧ B)

| (¬∧)

¬A, ¬B

Rule (¬∧) Corresponds to DeMorgan Law

$\neg(A \wedge B) \equiv (\neg A \cup \neg B)$

Negation Rule (¬¬)

I, ¬¬ A, J

| (¬¬)

I , A, J

¬¬A

| (¬¬)

A

Negation Rule (¬¬) Coresponds to

¬¬ A ≡ A

Transformation Rules :

(∧), (∪), (=>), (¬∧), (¬∪), (¬=>)

Transformation Rules Shorthand Form

(AUB) (U)

A, B

¬(A U B) (¬U)

¬A ¬B

(A ∧B) (∧)

A B

¬(A ∧ B) (¬∧)

¬A, ¬B

(A=>B) (=>)

¬A, B

¬(A => B) (¬=>)

A ¬B

¬¬A (¬¬)

A

+ Keep all Information

End when all leaves are literals

Example

- Let A be a Formula $(((a \Rightarrow \neg b) U c) \wedge (\neg a U \neg b))$
- **Find** CL_A
- $(((a \Rightarrow \neg b) U c) \wedge (\neg a U \neg b))$

$((a \Rightarrow \neg b) U c)$ $(\neg a U \neg b)$

$(a \Rightarrow \neg b), c$ $\neg a, b$ STOP

$\neg a, \neg b, c$ STOP

$$CL_A = \{\{\neg a, \neg b, c\}, \{\neg a, b\}\}$$

$$A \equiv CL_A$$

ARGUMENTS

- From (premises) $A_1, \ldots\ldots, A_n$ we conclude B

$$\frac{A_1, \ldots\ldots, A_n}{B}$$

Definition:

Argument $\frac{A_1, \ldots\ldots, A_n}{B}$ is **VALID** **iff**

$$\models ((A1 \wedge \ldots\ldots \wedge An) \Rightarrow B)$$

- Otherwise Argument is **NOT VALID**

ARGUMENTS

Valid Arguments ≡ Tautologically Valid

$A_1, \ldots\ldots., A_n, C$

can be formulas of Propositional or Predicate Language

Validity of Arguments

Remember: $\models A$ **iff** $=\mid \neg A$

Tautology (always true), Contradiction (always false)

This means that if we want to **decide** $\models A$ we **decide** $=\mid \neg A$ and **use** Resolution to do that

STEPS

Step 1: Negate A, i.e. take $\neg A$ and **find** the set of clauses corresponding to $\neg A$, i.e. **find** $\mathbf{CL}_{\{\neg A\}}$

Step 2: Use Completeness of Resolution

$\models A$ **iff** $\mathbf{CL}_{\{\neg A\}} \vdash_R \{\}$ i.e.

1. Look for a resolution deduction of $\{\}$ from $\mathbf{CL}_{\{\neg A\}}$
2. if YES – we have $\models A$
3. If there is no deduction of $\{\}$ we have: NOT $\models A$

Basic Theorems

T1. $=\mid CL$ **iff** $CL \vdash_R \{\}$

CL is inconsistent **iff** there is a resolution deduction of $\{\}$ from CL

T2. For any formula A, there is a set of clauses CL_A such that $A \equiv CL_A$

T3. $\models A$ **iff** $=\mid \neg A$

By **T2** we get that

$\models A$ **iff** $=\mid CL_{\{\neg A\}}$

And by **T1** and **T3** we get

T4. $\models A$ **iff** $CL_{\{\neg A\}} \vdash_R \{\}$

Exercise

- **Prove** By Propositional Resolution

$$\models (\neg(a \Rightarrow b) \Rightarrow (a \wedge \neg b))$$

Remember: $\models A$ **iff** $=\mid \neg A$ + use Resolution

Steps

Step 1: Find set of clauses corresponding to $\neg A$ i.e. find **CL**$_{\{\neg A\}}$

Step 2: Find deduction of $\{\}$ from **CL**$_{\{\neg A\}}$

i.e. show that **CL**$_{\{\neg A\}} \vdash_R \{\}$

DO IT!

Exercise Solution

- **Step 1:** Negate A and find the set of clauses for ¬A i.e. find **CL**$_{\{\neg A\}}$
- $\neg(\neg(a \Rightarrow b) \Rightarrow (a \wedge \neg b))$

$\neg(a \Rightarrow b)$ $\qquad\qquad$ $\neg(a \wedge \neg b)$

a $\qquad$ $\neg b$ $\qquad\qquad$ $\neg a, \neg\neg b$

$\qquad\qquad\qquad\qquad$ $\neg a, b$

Clauses: $\{a\}$ $\qquad$ $\{\neg b\}$ $\qquad$ $\{\neg a, b\}$

CL $_{\{\neg A\}} = \{\{a\}, \{\neg b\}, \{\neg a, b\}\}$

$\{b\}$

$\{\}$

Step 2: Check if **CL**$_{\{\neg A\}} \vdash_R \{\}$ – ***YES!***

Remark: NOT $\models A$ iff there is **no** deduction of $\{\}$ from **CL**$_{\{\neg A\}}$

Back To Arguments

- Use **resolution** to show that from $\mathbf{A_1,\ldots\ldots, A_n}$ we can deduce **B**

We **"can"** deduce B from $A_1,\ldots\ldots, A_n$ means **validity** of the argument

$$\frac{A_1,\ldots\ldots, A}{B}$$

This means that we have to show that

$$\models (A_1 \wedge \ldots\ldots \wedge A_n \Rightarrow B)$$

We have to use Resolution to prove that

$(A_1 \wedge \ldots\ldots \wedge A_n \Rightarrow B)$ is a tautology

Arguments

$\models (A_1 \wedge \ldots\ldots \wedge A_n \Rightarrow B)$ **iff**

$=\mid \neg (A_1 \wedge \ldots\ldots \wedge A_n \Rightarrow B)$ **iff**

$=\mid (A_1 \wedge \ldots\ldots \wedge A_n \wedge \neg B)$

- **Step 1:** we transform $(A_1 \wedge .. \wedge A_n \wedge \neg B)$ to clauses
- Take A1,......., An and **find**

$$CL_{A1}, \ldots, CL_{An}$$

and also **find** $CL_{\neg B}$ and then **form**

$$CL_{A1} \cup \ldots CL_{An} \cup CL_{\neg B} = CL$$

Step 2: examine whether $CL \vdash_R \{\}$

Remember

Argument $\frac{A1,\ldots\ldots,\ An}{B}$ is **valid**

iff $CL_{A1} \cup \ldots \cup CL_{An} \cup CL_{\neg B} \vdash_R \{\}$

↓

Argument is **not valid**

iff **never** $CL_{A1} \cup \ldots \cup CL_{An} \cup CL_{\neg B} \vdash_R \{\}$

We have some Resolution Strategies that allow us to cut down number of cases to consider

Example

Check if you can deduce

$B = (\neg(a \; U \; \neg b) \Rightarrow (\neg a \wedge b))$

from $A1 = ((a \Rightarrow \neg b) \Rightarrow a)$ and $A2 = (a \Rightarrow (b \Rightarrow a))$

Procedure:

1. **Find** $\mathbf{CL}_{\{A1\}}$, $\mathbf{CL}_{\{A2\}}$ and $CL_{\{\neg B\}}$
2. **Form** $\mathbf{CL} = \mathbf{CL}_{\{A1\}} \cup \mathbf{CL}_{\{A2\}} \cup \mathbf{CL}_{\{\neg B\}}$
3. **Check** if $\mathbf{CL} \vdash_R \{\}$ or if never $\mathbf{CL} \vdash_R \{\}$

Yes, we can

No, we can't

Example Solution

A1 = ((a => ¬b) => a)

¬(a=>¬b), a

a, a

¬¬b, a

b, a

We get: CL_{A1} = {{a}, { b, a}}

A2 = ((a=>(b=>a))

¬a , (b=>a)

¬a, ¬b, a

We get: CL_{A2} = {¬a, ¬b, a}

Example Solution

$\neg B = \neg(\neg(a \; U \; \neg b) \Rightarrow (\neg a \land b))$

$\neg(a \; U \; \neg b)$

$\neg(\neg a \land b)$

$\neg a$

$\neg\neg b$

$\neg\neg a, \neg b$

b

$a, \neg b$

CL = {{a}, {b, a}, {¬a, ¬b, a}, {¬a}, {b}, {a, ¬b}}

Remove Tautology Strategy gives us the set

CL = {{a}, {b, a}, {¬a}, {b}, {a, ¬b}}

Example Solution

- **CL** = {{a}, {b, a}, {¬a, ¬b, c}, {¬a}, {b}, {a, ¬b}}

{a} R on b

{}

Yes Argument is Valid

Next : Strategies for Resolution

Propositional Resolution
Part 3

Resolution Strategies

- **We present here some Deletion Strategies and discuss their Completeness.**

Deletion Strategies are restriction techniques in which clauses with specified properties are eliminated from set of clauses **CL** before they are used.

Pure Literals

Definition

A literal is **pure** in **CL** **iff** it **has no** complementary literal in any other clause in **CL**

Example: **CL** = { {a, b}, {¬ c, d}, {c,b}, {¬ d}}

a, b are **pure** and c, d, ¬ c, ¬ d are **not pure**

c has complement literal ¬ c in {¬ c, d} and

¬ c has complement literal c in {c,b}

d has a complement literal ¬d in the clause {¬ d} and

¬d has a complement literal d in {¬ c, d}

S1: Pure Literals Deletion Strategy

S1 Strategy: Remove all clauses that contain Pure Literals

Clauses that contain pure literals are useless for retention process.

One pure literal in a clause is enough for the clause removal

This Strategy is complete, i.e.

CL ⊢ {} iff CL' ⊢ {}

where **CL'** is obtained from **CL** by pure literal clauses **deletion**

Example

- **CL** = {{¬a, ¬b, c}, {¬p, d}, {¬b, d}, {a}, {b}, {¬c}}

  d, ¬p are pure,

CL' = {{¬a, ¬b, c}, {a}, {b}, {¬c}}

{¬b, c}

{c}

{}

S2. Tautology Deletion Strategy

- **Tautology –** a clause containing a **pair** of complementary literals (a and ¬a)
- **S2: Tautology Deletion:**

  CL' = Remove all Tautologies from **CL**
- Example:
- **CL** = {{ a, b, ¬a}, {b, ¬b, c}, {a}}

  CL' = {{a}}
- Tautology Deletion Strategy S2 is **COMPLETE**.

CL is satisfiable **≡** **CL'** **is** satisfiable

CL unsatisfiable **≡** **CL'** unsatisfiable

Exercise

- **Example:**
- **CL** = {{ a, ¬a, b}, {b, ¬b, c}} - remove tautologies- get

CL' with no elements, i.e. **CL'** = ϕ

CL is always satisfiable and so is **CL'** as Φ is always satisfiable!

Exercise

Prove correctness of Tautology Deletion Strategy

S3. Unit Resolution Strategy

- **A unit resolvent** – resolvent in which at least one of the parent clauses is **a unit clause** i.e. is a clause containing a single literal.
- **A unit deduction** – all derived clauses are unit resolvents.
- **A unit Refutation** – **unit deduction of the empty clause {}.**
- Example: {{a, b}, {¬a, c}, {¬b, c}, {¬c}}

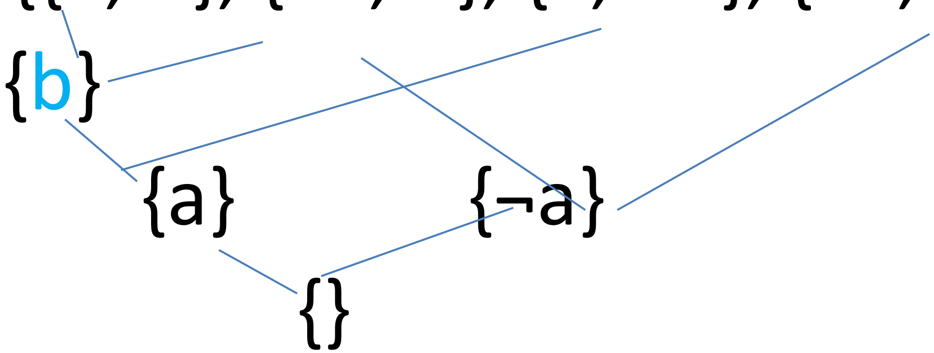
{¬a} {¬b}

{b}

{} **Efficient but not Complete!**

Unit Resolution not complete Example

- **CL** = {{a, b}, {¬a, b}, {a, ¬b}, {¬a, ¬b}}



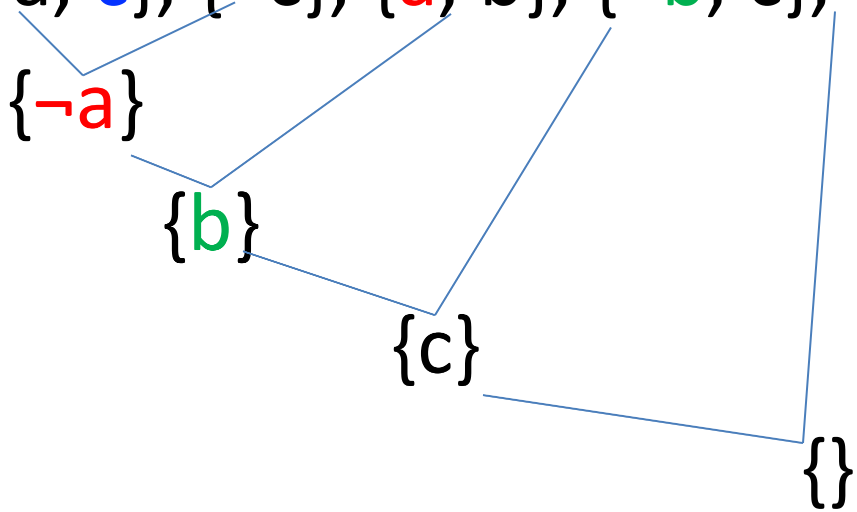
CL is **unsatisfiable**, but does not have unit deduction.

Horn Clause: a clause with at most one positive literal.

Theorem: Unit Resolution is complete on Horn Clauses.

Example of Unit Resolution Deduction

- **CL** = {{¬a, c}, {¬c}, {a, b}, {¬b, c}, {¬c}}



CL is not Horn but **CL⊢** {} by unit deduction.

Remark: if we get { } by unit deduction we are OK but if we don’t get { } by unit deduction it does not mean that **CL** is satisfiable, because unit strategy is not a Complete Strategy on non- Horn clauses.

S4. Input Resolution

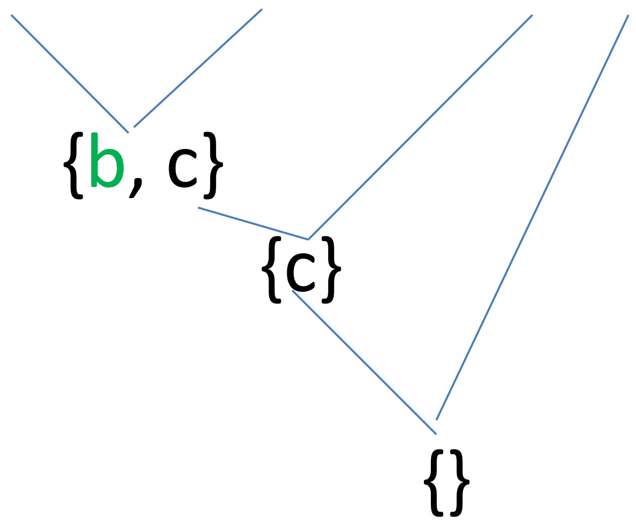
- **Input Resolution- At least one of the two parent clauses is in the initial database.**
- **Input Deduction-** all derived clauses are input resolvents
- **Input Refutation-** Input deduction of {}

THM 1: Unit and Input Resolution are equivalent.

THM 2: Input Resolution is complete only on Horn Clauses

Input Resolution Deduction

Example: **CL** = {{a, b}, {¬a, c}, {¬b, c}, {¬c}}



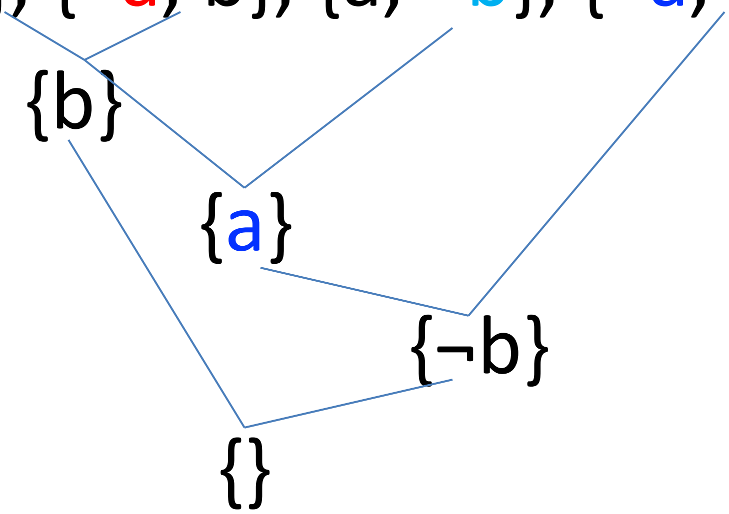
NOT Complete!

5. Linear Resolution

- **Linear Resolution** also called **Ancestry-Filtered** resolution is a slight generalization of Input Resolution.
- **A Linear Resolution: At least one of the parents is either in the initial DB or is in an Ancestor of the other parent.**
- **A Linear Deduction:** Uses only linear resolvents : each derived clauses is a linear resolvent
- **A Linear Refutation:** Linear deduction of { }.
- **Linear Resolution is complete**

Example

CL = {{a, b}, {¬a, b}, {a, ¬b}, {¬a, ¬ b}}



Here :

{a} is a parent of {¬b}

{b} is the ancestor of {¬b} (other parent of {¬b})

Linear Resolution

Linear Resolution is complete

There are also more modifications of the LR that are **complete**

Our Strategies work also for **Predicate Logic** Resolution

First papers

Kowalski 1974, 1976 “Logic for problem solving” “Predicate Logic as a programming language”.

Robinson 1965 “A Machinery Oriented logic based on the resolution principle” J Assoc. for Computing Machinery 12(1)