Chomsky and Greibach Normal Forms

Simplifying a CFG

- It is often convenient to simplify CFG

Simplifying a CFG

- It is often convenient to simplify CFG
- One of the simplest and most useful simplified forms of CFG is called the Chomsky normal form

Simplifying a CFG

- It is often convenient to simplify CFG
- One of the simplest and most useful simplified forms of CFG is called the Chomsky normal form
- Another normal form usually used in algebraic specifications is Greibach normal form

Simplifying a CFG

- It is often convenient to simplify CFG
- One of the simplest and most useful simplified forms of CFG is called the Chomsky normal form
- Another normal form usually used in algebraic specifications is Greibach normal form

Note the difference between grammar cleaning and simplifi cation

Note

Normal forms are useful when more advanced topics in computation theory are approached, as we shall see further

Definition

A context-free grammar G is in Chomsky normal form if every rule is of the form:

$$
\begin{aligned}
A &\longrightarrow BC \\
A &\longrightarrow a
\end{aligned}
$$

where a is a terminal, A, B, C are nonterminals, and B, C may not be the start variable (the axiom)

Note

The rule $S \longrightarrow \epsilon$, where S is the start variable, is not excluded from a CFG in Chomsky normal form.

Theorem 2.9

Any context-free language is generated by a context-free grammar in Chomsky normal form.

Theorem 2.9

Any context-free language is generated by a context-free grammar in Chomsky normal form.

Proof idea:

Theorem 2.9

Any context-free language is generated by a context-free grammar in Chomsky normal form.

Proof idea:

- Show that any CFG G can be converted into a CFG G' in Chomsky normal form

Theorem 2.9

Any context-free language is generated by a context-free grammar in Chomsky normal form.

Proof idea:

- Show that any CFG G can be converted into a CFG G' in Chomsky normal form
- Conversion procedure has several stages where the rules that violate Chomsky normal form conditions are replaced with equivalent rules that satisfy these conditions

Theorem 2.9

Any context-free language is generated by a context-free grammar in Chomsky normal form.

Proof idea:

- Show that any CFG G can be converted into a CFG G' in Chomsky normal form
- Conversion procedure has several stages where the rules that violate Chomsky normal form conditions are replaced with equivalent rules that satisfy these conditions
- Order of transformations: (1) add a new start variable, (2) eliminate all ϵ-rules, (3) eliminate unit-rules, (4) convert other rules

Theorem 2.9

Any context-free language is generated by a context-free grammar in Chomsky normal form.

Proof idea:

- Show that any CFG G can be converted into a CFG G' in Chomsky normal form
- Conversion procedure has several stages where the rules that violate Chomsky normal form conditions are replaced with equivalent rules that satisfy these conditions
- Order of transformations: (1) add a new start variable, (2) eliminate all ϵ-rules, (3) eliminate unit-rules, (4) convert other rules
- Check that the obtained CFG G' defi nes the same language

Proof

Let $G = (N, T, R, S)$ be the original CFG.

Proof

Let $G = (N, T, R, S)$ be the original CFG.

Step 1: add a new start symbol S_0 to N, and the rule $S_0 \longrightarrow S$ to R

Proof

Let $G = (N, T, R, S)$ be the original CFG.

Step 1: add a new start symbol S_0 to N, and the rule $S_0 \longrightarrow S$ to R

Note: this change guarantees that the start symbol of G' does not occur on the rhs of any rule

Step 2: eliminate ϵ-rules

Repeat

1. Eliminate the ϵ rule $A \longrightarrow \epsilon$ from R where A is not the start symbol
2. For each occurrence of A on the rhs of a rule, add a new rule to R with that occurrence of A deleted
   Example: replace $B \longrightarrow uAv$ by $B \longrightarrow uAv|uv$;
   replace $B \longrightarrow uAvAw$ by $B \longrightarrow uAvAw|uvAw|aAvw|uvw$
3. Replace the rule $B \longrightarrow A$, (if it is present) by $B \longrightarrow A|\epsilon$ unless the rule $B \longrightarrow \epsilon$ has been previously eliminated

until all ϵ rules are eliminated

Step 3: remove unit rules

Repeat

1. Remove a unit rule $A \longrightarrow B \in R$
2. For each rule $B \longrightarrow u \in R$, add the rule $A \longrightarrow u$ to R, unless $B \to u$ was a unit rule previously removed

until all unit rules are eliminated

Note: u is a string of variables and terminals

Convert all remaining rules

Repeat

1. Replace a rule $A \longrightarrow u_1u_2 \ldots u_k$, $k \geq 3$, where each u_i, $1 \leq i \leq k$, is a variable or a terminal, by:
   $A \longrightarrow u_1A_1$, $A_1 \longrightarrow u_2A_2$, ..., $A_{k-2} \longrightarrow u_{k-1}u_k$
   where A_1,A_2, ..., A_{k-2} are new variables
2. If $k \geq 2$ replace any terminal u_i with a new variable U_i and add the rule $U_i \longrightarrow u_i$

until no rules of the form $A \longrightarrow u_1u_2 \ldots u_k$ with $k \geq 3$ remain

Convert all remaining rules

Repeat

1. Replace a rule $A \longrightarrow u_1 u_2 \ldots u_k$, $k \geq 3$, where each u_i, $1 \leq i \leq k$, is a variable or a terminal, by:
   $A \longrightarrow u_1 A_1$, $A_1 \longrightarrow u_2 A_2$, ..., $A_{k-2} \longrightarrow u_{k-1} u_k$
   where A_1,A_2, ..., A_{k-2} are new variables
2. If $k \geq 2$ replace any terminal u_i with a new variable U_i and add the rule $U_i \longrightarrow u_i$

until no rules of the form $A \longrightarrow u_1 u_2 \ldots u_k$ with $k \geq 3$ remain

Convert all remaining rules

Repeat

1. Replace a rule $A \longrightarrow u_1 u_2 \ldots u_k$, $k \geq 3$, where each u_i, $1 \leq i \leq k$, is a variable or a terminal, by:
   $A \longrightarrow u_1 A_1$, $A_1 \longrightarrow u_2 A_2$, . . ., $A_{k-2} \longrightarrow u_{k-1} u_k$
   where A_1,A_2, . . ., A_{k-2} are new variables
2. If $k \geq 2$ replace any terminal u_i with a new variable U_i and add the rule $U_i \longrightarrow u_i$

until no rules of the form $A \longrightarrow u_1 u_2 \ldots u_k$ with $k \geq 3$ remain

Example CFG conversion

Consider the grammar G_6 whose rules are:

$$\begin{array}{lcl} S & \longrightarrow & ASA|aB \\ A & \longrightarrow & B|S \\ B & \longrightarrow & b|\epsilon \end{array}$$

Notation: symbols removed are green and those added are red.

After fi rst step of transformation we get:

$$\begin{array}{lcl} S_0 & \longrightarrow & S \\ S & \longrightarrow & ASA|aB \\ A & \longrightarrow & B|S \\ B & \longrightarrow & b|\epsilon \end{array}$$

Example CFG conversion

Consider the grammar G_6 whose rules are:

$$\begin{array}{rcl} S & \longrightarrow & ASA|aB \\ A & \longrightarrow & B|S \\ B & \longrightarrow & b|\epsilon \end{array}$$

Notation: symbols removed are green and those added are red.

After first step of transformation we get:

$$\begin{array}{rcl} S_0 & \longrightarrow & S \\ S & \longrightarrow & ASA|aB \\ A & \longrightarrow & B|S \\ B & \longrightarrow & b|\epsilon \end{array}$$

Example CFG conversion

Consider the grammar G_6 whose rules are:

$$\begin{array}{rcl} S & \longrightarrow & ASA|aB \\ A & \longrightarrow & B|S \\ B & \longrightarrow & b|\epsilon \end{array}$$

Notation: symbols removed are green and those added are red.

After first step of transformation we get:

$$\begin{array}{rcl} S_0 & \longrightarrow & S \\ S & \longrightarrow & ASA|aB \\ A & \longrightarrow & B|S \\ B & \longrightarrow & b|\epsilon \end{array}$$

Removing ϵ rules

Removing $B \to \epsilon$:

$$\begin{array}{rcl} S_0 & \longrightarrow & S \\ S & \longrightarrow & ASA|aB|a \\ A & \longrightarrow & B|S|\epsilon \\ B & \longrightarrow & b|\epsilon \end{array}$$

Removing $A \to \epsilon$:

$$\begin{array}{rcl} S_0 & \longrightarrow & S \\ S & \longrightarrow & ASA|aB|a|SA|AS|S \\ A & \longrightarrow & B|S|\epsilon \\ B & \longrightarrow & b \end{array}$$

Removing ϵ rules

Removing $B \to \epsilon$:

$$
\begin{aligned}
S_0 &\longrightarrow S \\
S &\longrightarrow ASA|aB|a \\
A &\longrightarrow B|S|\epsilon \\
B &\longrightarrow b|\epsilon
\end{aligned}
$$

Removing $A \to \epsilon$:

$$
\begin{aligned}
S_0 &\longrightarrow S \\
S &\longrightarrow ASA|aB|a|SA|AS|S \\
A &\longrightarrow B|S|\epsilon \\
B &\longrightarrow b
\end{aligned}
$$

Removing ϵ rules

Removing $B \to \epsilon$:

$$\begin{aligned} S_0 &\longrightarrow S \\ S &\longrightarrow ASA|aB|a \\ A &\longrightarrow B|S|\epsilon \\ B &\longrightarrow b|\epsilon \end{aligned}$$

Removing $A \to \epsilon$:

$$\begin{aligned} S_0 &\longrightarrow S \\ S &\longrightarrow ASA|aB|a|SA|AS|S \\ A &\longrightarrow B|S|\epsilon \\ B &\longrightarrow b \end{aligned}$$

Removing unit rule

Removing $S \to S$:

$$
\begin{aligned}
S_0 &\longrightarrow S \\
S &\longrightarrow ASA|aB|a|SA|AS|S \\
A &\longrightarrow B|S \\
B &\longrightarrow b
\end{aligned}
$$

Removing $S_0 \to S$:

$$
\begin{aligned}
S_0 &\longrightarrow S|ASA|aB|a|SA|AS \\
S &\longrightarrow ASA|aB|a|SA|AS \\
A &\longrightarrow B|S \\
B &\longrightarrow b
\end{aligned}
$$

Removing unit rule

Removing $S \to S$:

$$\begin{aligned} S_0 &\longrightarrow S \\ S &\longrightarrow ASA|aB|a|SA|AS|S \\ A &\longrightarrow B|S \\ B &\longrightarrow b \end{aligned}$$

Removing $S_0 \to S$:

$$\begin{aligned} S_0 &\longrightarrow S|ASA|aB|a|SA|AS \\ S &\longrightarrow ASA|aB|a|SA|AS \\ A &\longrightarrow B|S \\ B &\longrightarrow b \end{aligned}$$

Removing unit rule

Removing $S \to S$:

$$\begin{array}{rcl} S_0 & \longrightarrow & S \\ S & \longrightarrow & ASA|aB|a|SA|AS|S \\ A & \longrightarrow & B|S \\ B & \longrightarrow & b \end{array}$$

Removing $S_0 \to S$:

$$\begin{array}{rcl} S_0 & \longrightarrow & S|ASA|aB|a|SA|AS \\ S & \longrightarrow & ASA|aB|a|SA|AS \\ A & \longrightarrow & B|S \\ B & \longrightarrow & b \end{array}$$

More unit rules

Removing $A \to B$:

$$\begin{aligned} S_0 &\longrightarrow ASA|aB|a|SA|AS \\ S &\longrightarrow ASA|aB|a|SA|AS \\ A &\longrightarrow B|S|b \\ B &\longrightarrow b \end{aligned}$$

Removing $A \to S$:

$$\begin{aligned} S_0 &\longrightarrow ASA|aB|a|SA|AS \\ S &\longrightarrow ASA|aB|a|SA|AS \\ A &\longrightarrow S|b|ASA|aB|a|SA|AS \\ B &\longrightarrow b \end{aligned}$$

More unit rules

Removing $A \rightarrow B$:

$$\begin{array}{rcl} S_0 & \longrightarrow & ASA|aB|a|SA|AS \\ S & \longrightarrow & ASA|aB|a|SA|AS \\ A & \longrightarrow & B|S|b \\ B & \longrightarrow & b \end{array}$$

Removing $A \rightarrow S$:

$$\begin{array}{rcl} S_0 & \longrightarrow & ASA|aB|a|SA|AS \\ S & \longrightarrow & ASA|aB|a|SA|AS \\ A & \longrightarrow & S|b|ASA|aB|a|SA|AS \\ B & \longrightarrow & b \end{array}$$

More unit rules

Removing $A \rightarrow B$:

$$\begin{aligned} S_0 &\longrightarrow ASA|aB|a|SA|AS \\ S &\longrightarrow ASA|aB|a|SA|AS \\ A &\longrightarrow B|S|b \\ B &\longrightarrow b \end{aligned}$$

Removing $A \rightarrow S$:

$$\begin{aligned} S_0 &\longrightarrow ASA|aB|a|SA|AS \\ S &\longrightarrow ASA|aB|a|SA|AS \\ A &\longrightarrow S|b|ASA|aB|a|SA|AS \\ B &\longrightarrow b \end{aligned}$$

Converting remaining rules

$$
\begin{aligned}
S_0 &\longrightarrow AA_1|UB|a|SA|AS \\
S &\longrightarrow AA_1|UB|a|SA|AS \\
A &\longrightarrow b|AA_1|UB|a|SA|AS \\
A_1 &\longrightarrow SA \\
U &\longrightarrow a \\
B &\longrightarrow b
\end{aligned}
$$

Converting remaining rules

$$\begin{aligned}
S_0 &\longrightarrow AA_1|UB|a|SA|AS \\
S &\longrightarrow AA_1|UB|a|SA|AS \\
A &\longrightarrow b|AA_1|UB|a|SA|AS \\
A_1 &\longrightarrow SA \\
U &\longrightarrow a \\
B &\longrightarrow b
\end{aligned}$$

Converting remaining rules

$$
\begin{aligned}
S_0 &\longrightarrow AA_1|UB|a|SA|AS \\
S &\longrightarrow AA_1|UB|a|SA|AS \\
A &\longrightarrow b|AA_1|UB|a|SA|AS \\
A_1 &\longrightarrow SA \\
U &\longrightarrow a \\
B &\longrightarrow b
\end{aligned}
$$

Note

- The conversion procedure produces several variables U_i along with several rules $U_i \to a$.
- Since all these represent the same rule, we may simplify the result using a single variable U and a single rule $U \to a$

Note

- The conversion procedure produces several variables U_i along with several rules $U_i \to a$.
- Since all these represent the same rule, we may simplify the result using a single variable U and a single rule $U \to a$

Note

- The conversion procedure produces several variables U_i along with several rules $U_i \to a$.
- Since all these represent the same rule, we may simplify the result using a single variable U and a single rule $U \to a$

Greibach Normal Form

A context-free grammar $G = (V, \Sigma, R, S)$ is in Greibach normal form if each rule $r \in R$ has the property: $lhs(r) \in V$, $rhs(r) = a\alpha$, $a \in \Sigma$ and $\alpha \in V^*$.

Note: Greibach normal form provides a justifi cation of operator prefi x-notation usually employed in algebra.

Greibach Normal Form

A context-free grammar $G = (V, \Sigma, R, S)$ is in Greibach normal form if each rule $r \in R$ has the property: $lhs(r) \in V$, $rhs(r) = a\alpha$, $a \in \Sigma$ and $\alpha \in V^*$.

Note: Greibach normal form provides a justifi cation of operator prefi x-notation usually employed in algebra.

Greibach Normal Form

A context-free grammar $G = (V, \Sigma, R, S)$ is in Greibach normal form if each rule $r \in R$ has the property: $lhs(r) \in V$, $rhs(r) = a\alpha$, $a \in \Sigma$ and $\alpha \in V^*$.

Note: Greibach normal form provides a justifi cation of operator prefi x-notation usually employed in algebra.

Greibach Theorem

Every CFL L where $\epsilon \notin L$ can be generated by a CFG in Greibach normal form.

Proof idea: Let $G = (V, \Sigma, R, S)$ be a CFG generating L. Assume that G is in Chomsky normal form

- Let $V = \{A_1, A_2, \ldots, A_m\}$ be an ordering of nonterminals.
- Construct the Greibach normal form from Chomsky normal form

Greibach Theorem

Every CFL L where $\epsilon \notin L$ can be generated by a CFG in Greibach normal form.

Proof idea: Let $G = (V, \Sigma, R, S)$ be a CFG generating L. Assume that G is in Chomsky normal form

- Let $V = \{A_1, A_2, \ldots, A_m\}$ be an ordering of nonterminals.
- Construct the Greibach normal form from Chomsky normal form

Greibach Theorem

Every CFL L where $\epsilon \notin L$ can be generated by a CFG in Greibach normal form.

Proof idea: Let $G = (V, \Sigma, R, S)$ be a CFG generating L. Assume that G is in Chomsky normal form

- Let $V = \{A_1, A_2, \ldots, A_m\}$ be an ordering of nonterminals.
- Construct the Greibach normal form from Chomsky normal form

Construction

1. Modify the rules in R so that if $A_i \to A_j\gamma \in R$ then $j > i$
2. Starting with A_1 and proceeding to A_m this is done as follows:
   (a) Assume that productions have been modifi ed so that for $1 \leq i \leq k$, $A_i \to A_j\gamma \in R$ only if $j > i$
   (b) If $A_k \to A_j\gamma$ is a production with $j < k$, generate a new set of productions substituting for the A_j the rhs of each A_j production
   (c) Repeating (b) at most $k - 1$ times we obtain rules of the form $A_k \to A_p\gamma, p \geq k$
   (d) Replace rules $A_k \to A_k\gamma$ by removing left-recursive rules

Construction

1. Modify the rules in R so that if $A_i \to A_j\gamma \in R$ then $j > i$
2. Starting with A_1 and proceeding to A_m this is done as follows:
   (a) Assume that productions have been modifi ed so that for $1 \leq i \leq k$, $A_i \to A_j\gamma \in R$ only if $j > i$
   (b) If $A_k \to A_j\gamma$ is a production with $j < k$, generate a new set of productions substituting for the A_j the rhs of each A_j production
   (c) Repeating (b) at most $k - 1$ times we obtain rules of the form $A_k \to A_p\gamma$, $p \geq k$
   (d) Replace rules $A_k \to A_k\gamma$ by removing left-recursive rules

Construction

1. Modify the rules in R so that if $A_i \to A_j\gamma \in R$ then $j > i$
2. Starting with A_1 and proceeding to A_m this is done as follows:
   (a) Assume that productions have been modifi ed so that for $1 \leq i \leq k$, $A_i \to A_j\gamma \in R$ only if $j > i$
   (b) If $A_k \to A_j\gamma$ is a production with $j < k$, generate a new set of productions substituting for the A_j the rhs of each A_j production
   (c) Repeating (b) at most $k - 1$ times we obtain rules of the form $A_k \to A_p\gamma$, $p \geq k$
   (d) Replace rules $A_k \to A_k\gamma$ by removing left-recursive rules

Removing left-recursion

Left-recursion can be eliminated by the following scheme:

- If $A \to A\alpha_1|A\alpha_2 \ldots |A\alpha_r$ are all A left recursive rules, and $A \to \beta_1|\beta_2|\ldots|\beta_s$ are all remaining A-rules then chose a new nonterminal, say B
- Add the new B-rules $B \to \alpha_i|\alpha_i B$, $1 \leq i \leq r$
- Replace the A-rules by $A \to \beta_i|\beta_i B$, $1 \leq i \leq s$

This construction preserve the language L.

Removing left-recursion

Left-recursion can be eliminated by the following scheme:

- If $A \rightarrow A\alpha_1|A\alpha_2 \ldots |A\alpha_r$ are all A left recursive rules, and $A \rightarrow \beta_1|\beta_2| \ldots |\beta_s$ are all remaining A-rules then chose a new nonterminal, say B
- Add the new B-rules $B \rightarrow \alpha_i|\alpha_i B$, $1 \leq i \leq r$
- Replace the A-rules by $A \rightarrow \beta_i|\beta_i B$, $1 \leq i \leq s$

This construction preserve the language L.

Removing left-recursion

Left-recursion can be eliminated by the following scheme:

- If $A \to A\alpha_1 | A\alpha_2 \ldots | A\alpha_r$ are all A left recursive rules, and $A \to \beta_1 | \beta_2 | \ldots | \beta_s$ are all remaining A-rules then chose a new nonterminal, say B
- Add the new B-rules $B \to \alpha_i | \alpha_i B$, $1 \leq i \leq r$
- Replace the A-rules by $A \to \beta_i | \beta_i B$, $1 \leq i \leq s$

This construction preserve the language L.

More on Greibach NF

See Introduction to Automata Theory, Languages, and Computation, J.E, Hopcroft and J.D Ullman, Addison-Wesley 1979, p. 94–96

Example

Convert the CFG

$G = (\{A_1, A_2, A_3\}, \{a, b\}, R, A_1)$

where

$R = \{A_1 \to A_2A_3, A_2 \to A_3A_1|b, A_3 \to A_1A_2|a\}$

into Greibach normal form.

Example

Convert the CFG

$G = (\{A_1, A_2, A_3\}, \{a, b\}, R, A_1)$

where

$R = \{A_1 \rightarrow A_2A_3, A_2 \rightarrow A_3A_1|b, A_3 \rightarrow A_1A_2|a\}$

into Greibach normal form.

Example

Convert the CFG

$G = (\{A_1, A_2, A_3\}, \{a, b\}, R, A_1)$

where

$R = \{A_1 \to A_2A_3, A_2 \to A_3A_1|b, A_3 \to A_1A_2|a\}$

into Greibach normal form.

Solution

1. *Step 1:* ordering the rules: (Only A_3 rules violate ordering conditions, hence only A_3 rules need to be changed). Following the procedure we replace A_3 rules by: $A_3 \to A_3A_1A_3A_2|bA_3A_2|a$
2. Eliminating left-recursion we get: $A_3 \to bA_3A_2B_3|aB_3|bA_3A_2|a$, $B_3 \to A_1A_3A_2|A_1A_3A_2B_3$
3. All A_3 rules start with a terminal. We use them to replace $A_1 \to A_2A_3$. This introduces the rules $B_3 \to A_1A_3A_2|A_1A_3A_2B_3$
4. Use A_1 production to make them start with a terminal

Solution

1. *Step 1:* ordering the rules: (Only A_3 rules violate ordering conditions, hence only A_3 rules need to be changed).
   Following the procedure we replace A_3 rules by:
   $A_3 \rightarrow A_3A_1A_3A_2|bA_3A_2|a$
2. Eliminating left-recursion we get: $A_3 \rightarrow bA_3A_2B_3|aB_3|bA_3A_2|a$, $B_3 \rightarrow A_1A_3A_2|A_1A_3A_2B_3$
3. All A_3 rules start with a terminal. We use them to replace $A_1 \rightarrow A_2A_3$. This introduces the rules $B_3 \rightarrow A_1A_3A_2|A_1A_3A_2B_3$
4. Use A_1 production to make them start with a terminal

Solution

1. *Step 1:* ordering the rules: (Only A_3 rules violate ordering conditions, hence only A_3 rules need to be changed).
   Following the procedure we replace A_3 rules by:
   $A_3 \rightarrow A_3A_1A_3A_2|bA_3A_2|a$
2. Eliminating left-recursion we get: $A_3 \rightarrow bA_3A_2B_3|aB_3|bA_3A_2|a$,
   $B_3 \rightarrow A_1A_3A_2|A_1A_3A_2B_3$
3. All A_3 rules start with a terminal. We use them to replace $A_1 \rightarrow A_2A_3$. This introduces the rules $B_3 \rightarrow A_1A_3A_2|A_1A_3A_2B_3$
4. Use A_1 production to make them start with a terminal