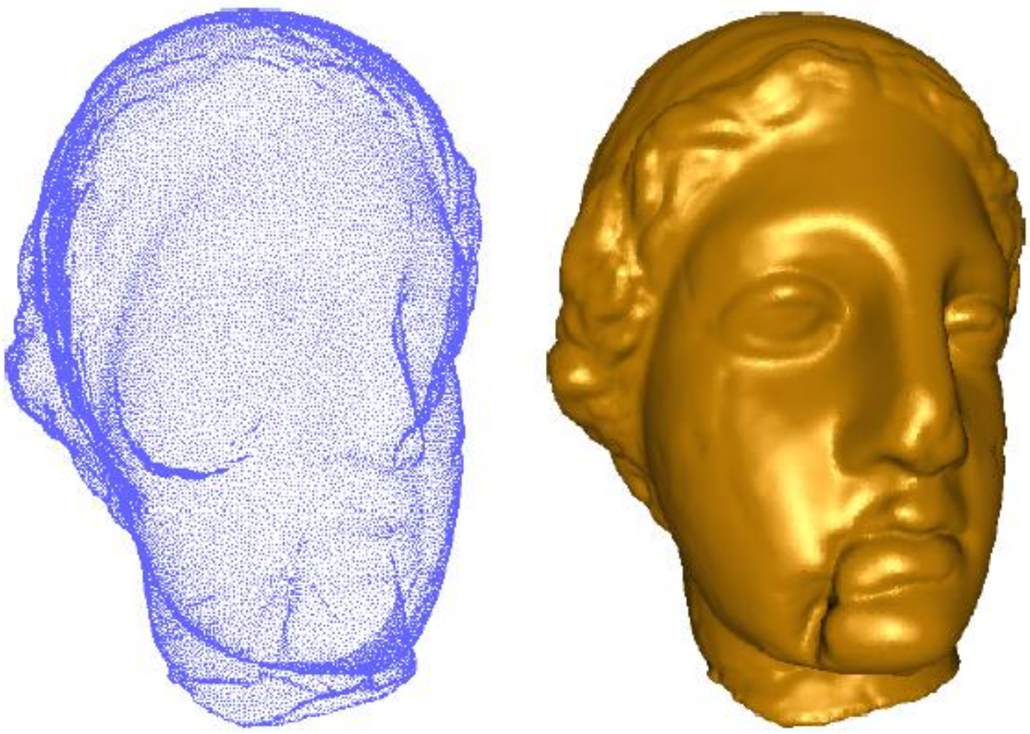
Least Squares Approach for Computer Graphics (From Point Cloud to CAD Models - A Brief Introduction)

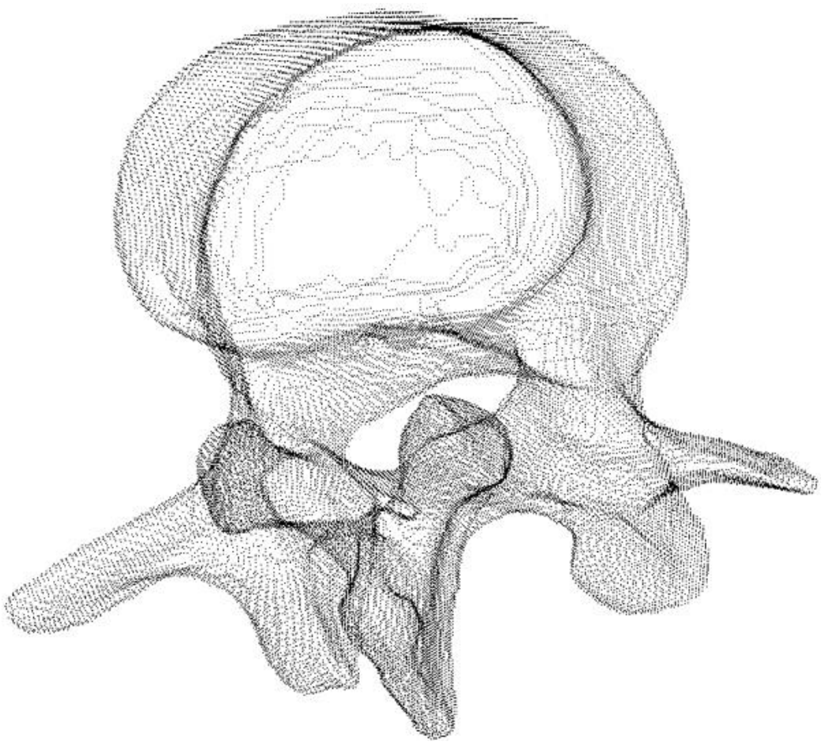
Motivation

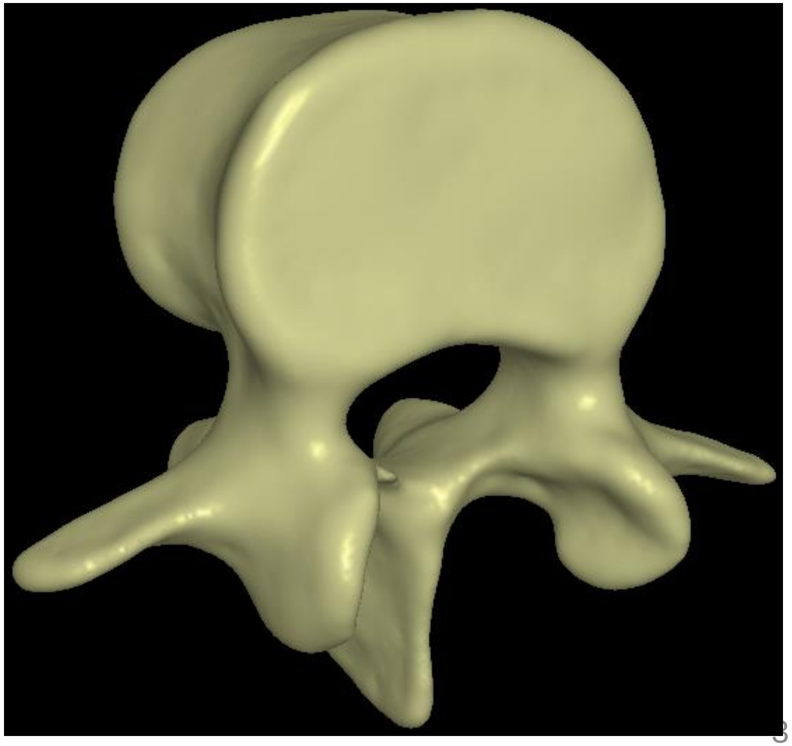
- From 3D points to CAD models
- Local surface fitting to 3D points



Reverse Engineering

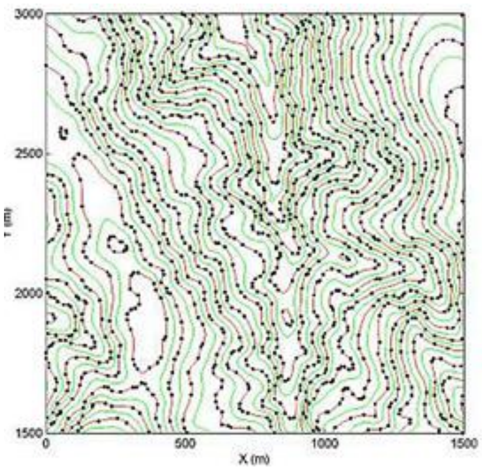
- From physical prototypes to digital prototypes via reverse engineering

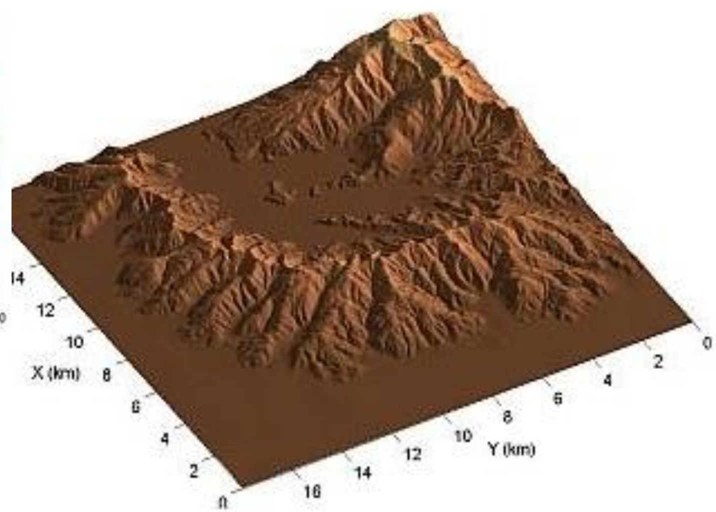


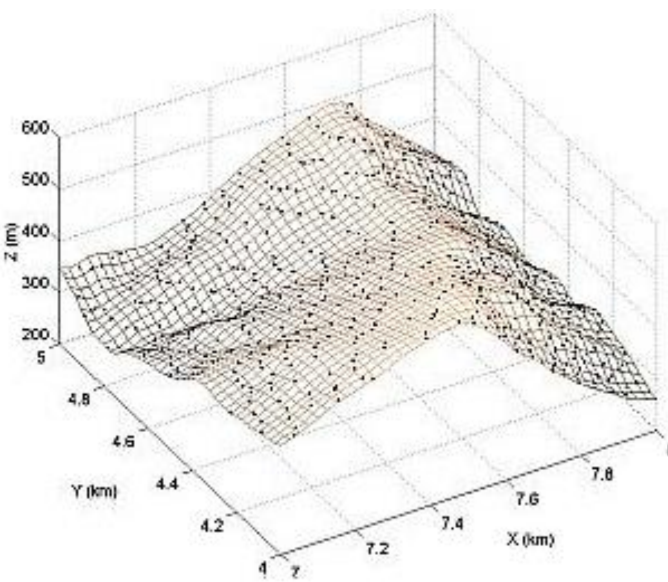


2D Terrain Modeling

- A simplified case

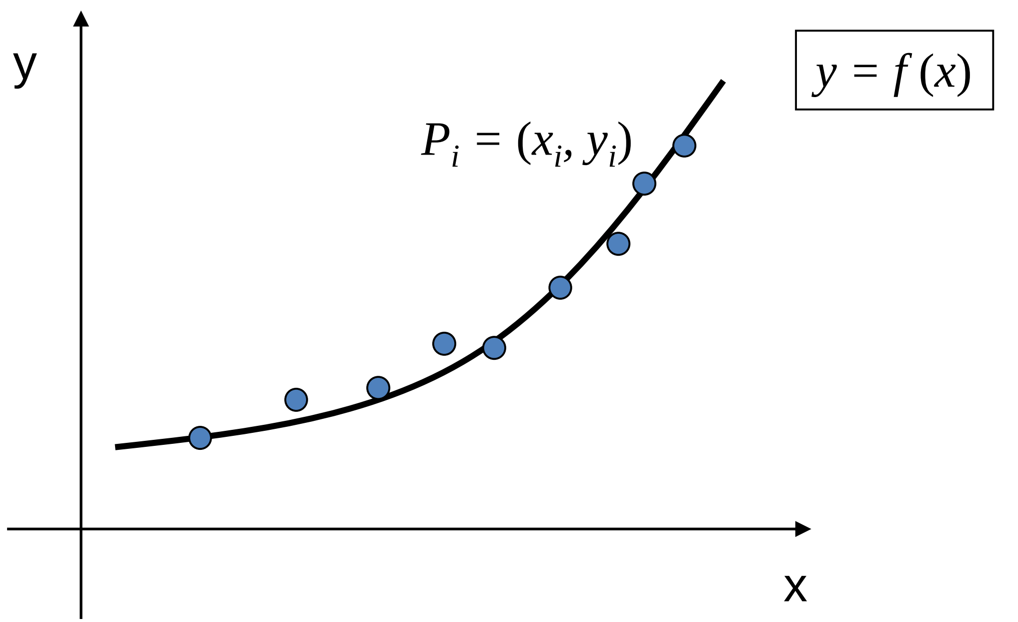






Motivation

- Given data points, fit a function that is "close" to the points

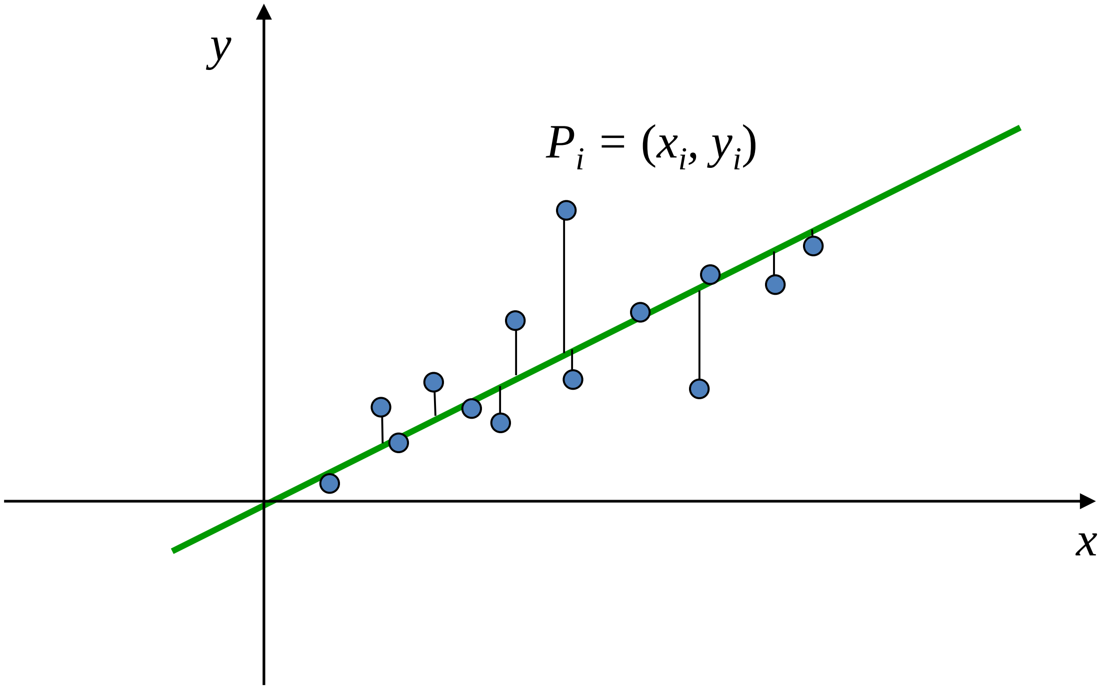


Outline

- Least squares approach
  - General / Polynomial fitting
  - Linear systems of equations
  - Local polynomial surface fitting

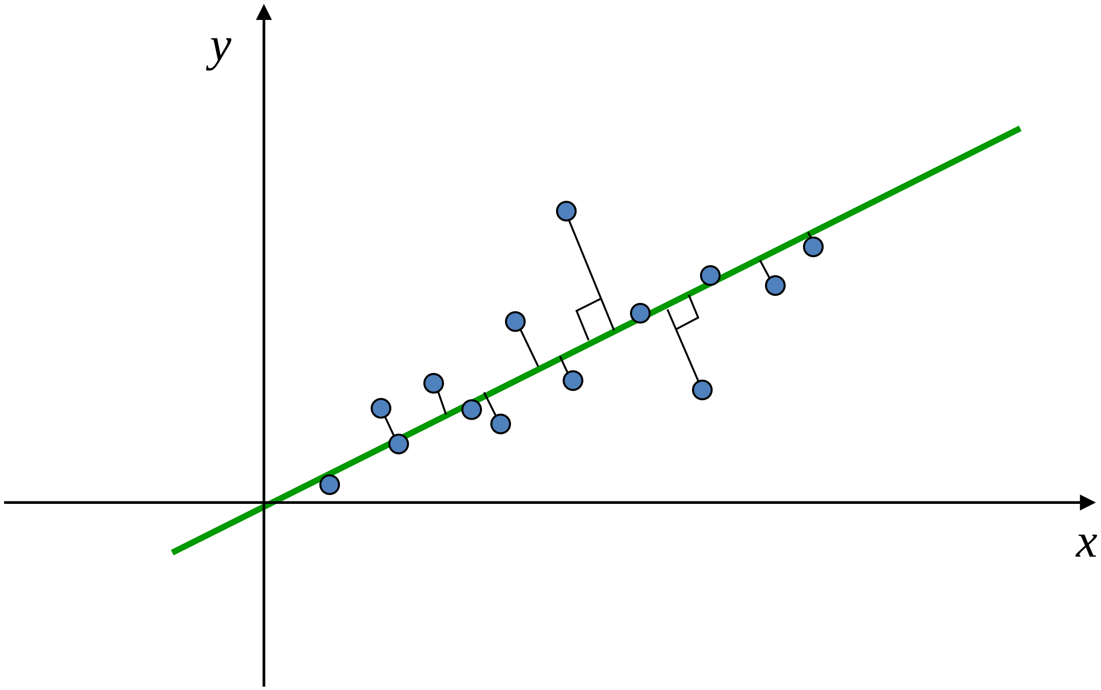
Line Fitting

- y-offset minimization



Line Fitting

- Orthogonal offset minimization – Principal Component Analysis (PCA)



Line Fitting

- Find a line $y = ax + b$ that minimizes

$$E(a,b) = \sum_{i=1}^{n} [y_i - (ax_i + b)]^2$$

- $E(a,b)$ is quadratic in the unknown parameters a, b

- Another option would be, for example:

$$AbsErr(a,b) = \sum_{i=1}^{n} \left| y_i - (ax_i + b) \right|$$

- But – it is not differentiable, harder to minimize…

Line Fitting – LS Minimization

- To find optimal a, b we differentiate $E(a, b)$:

$$\frac{\partial}{\partial a} E(a,b) = \sum_{i=1}^{n} (-2\mathrm{x}_\mathrm{i})[\mathrm{y}_\mathrm{i} - (\mathrm{a}\mathrm{x}_\mathrm{i} + \mathrm{b})] = 0$$

$$\frac{\partial}{\partial b} E(a,b) = \sum_{i=1}^{n} (-2)[\mathrm{y}_\mathrm{i} - (\mathrm{a}\mathrm{x}_\mathrm{i} + \mathrm{b})] = 0$$

Line Fitting – LS Minimization

- We obtain two linear equations for a, b:

$$\sum_{i=1}^{n} (-2x_i)[y_i - (ax_i + b)] = 0$$

$$\sum_{i=1}^{n} (-2)[y_i - (ax_i + b)] = 0$$

Line Fitting – LS Minimization

- We obtain two linear equations for a, b:

$$(1) \quad \sum_{i=1}^{n} [x_i y_i - a x_i^2 - b x_i] = 0$$

$$(2) \quad \sum_{i=1}^{n} [y_i - a x_i - b] = 0$$

Line Fitting – LS Minimization

- We obtain two linear equations

$$(\sum_{i=1}^{n} x_i^2)a + (\sum_{i=1}^{n} x_i)b = \sum_{i=1}^{n} x_i y_i$$

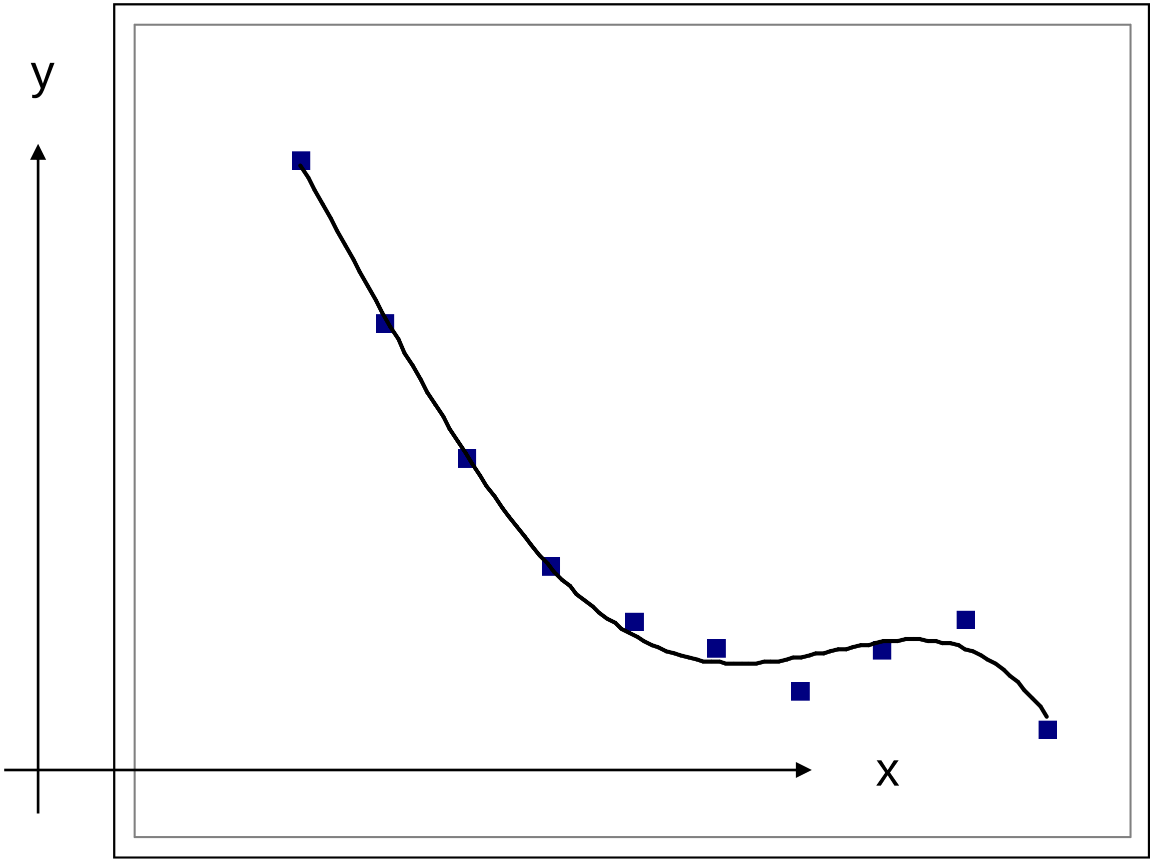
$$(\sum_{i=1}^{n} x_i)a + (\sum_{i=1}^{n} 1)b = \sum_{i=1}^{n} y_i$$

Line Fitting – LS Minimization

- Solve for a, b using (for example) Gauss elimination
- Question: why the solution is the *minimum* for the error function?

$$E(a, b) = \sum_{i=1}^{n} [y_i - (ax_i + b)]^2$$

Fitting Polynomials



Fitting Polynomials

- Decide on the degree of the polynomial, k
- Want to fit $f(x) = a_k x^k + a_{k-1} x^{k-1} + \ldots + a_1 x + a_0$
- Minimize:

$$E(a_0, a_1, \ldots, a_k) = \sum_{i=1}^{n} [y_i - (a_k x_i^k + a_{k-1} x_i^{k-1} + \ldots + a_1 x_i + a_0)]^2$$

$$\frac{\partial}{\partial a_m} E(a_0, \ldots, a_k) = \sum_{i=1}^{n} (-2x^m)[y_i - (a_k x_i^k + a_{k-1} x_i^{k-1} + \ldots + a_0)] = 0$$

Fitting Polynomials

- We obtain a linear system of $\mathrm{k}+1$ in $\mathrm{k}+1$ variables

$$
\begin{pmatrix}
\sum_{i=1}^{n} 1 & \sum_{i=1}^{n} x_i & \cdots & \sum_{i=1}^{n} x_i^{\,k} \\
\sum_{i=1}^{n} x_i & \sum_{i=1}^{n} x_i^{\,2} & \cdots & \sum_{i=1}^{n} x_i^{\,k+1} \\
\vdots & \vdots & \ddots & \vdots \\
\sum_{i=1}^{n} x_i^{\,k} & \sum_{i=1}^{n} x_i^{\,k+1} & \cdots & \sum_{i=1}^{n} x_i^{\,2k}
\end{pmatrix}
\begin{pmatrix}
a_0 \\ a_1 \\ \vdots \\ a_k
\end{pmatrix}
=
\begin{pmatrix}
\sum_{i=1}^{n} 1 \cdot y_i \\
\sum_{i=1}^{n} x_i y_i \\
\\
\sum_{i=1}^{n} x_i^{\,k} y_i
\end{pmatrix}
$$

General Parametric Fitting

- We can use this approach to fit any function $f(x)$
  - Specified by parameters $a, b, c, \ldots$
  - The expression $f(x)$ linearly depends on the parameters $a, b, c, \ldots$

General Parametric Fitting

- Want to fit function $f_{abc\ldots}(x)$ to data points (x_i, y_i)
  - Define $E(a,b,c,\ldots) = \sum_{i=1}^{n}[y_i - f_{abc\ldots}(x_i)]^2$
  - Solve the linear system

$$\frac{\partial}{\partial a}E(a,b,c,\ldots) = \sum_{i=1}^{n}(-2\frac{\partial}{\partial a}f_{abc\ldots}(x_i))[y_i - f(x_i)] = 0$$

$$\frac{\partial}{\partial b}E(a,b,c,\ldots) = \sum_{i=1}^{n}(-2\frac{\partial}{\partial b}f_{abc\ldots}(x_i))[y_i - f(x_i)] = 0$$

$$\vdots$$

General Parametric Fitting

- It can even be some crazy function like

$$f(x) = \lambda_1 \sin^2 x + \lambda_2 \ \mathbf{e}^{-\frac{x^2}{\sqrt{2\pi}}} + \lambda_3 \ x^{17}$$

- Or in general:

$$f_{\lambda_1, \lambda_1, \ldots, \lambda_k}(x) = \lambda_1 f_1(x) + \lambda_2 f_2(x) + \ldots + \lambda_k f_k(x)$$

Solving Linear Systems in LS Sense

- Let's look at the problem a little differently:
  - We have data points (x_i, y_i)
  - We want the function $f(x)$ to go *through* the points:

$$\forall\ i = 1, \ldots, n: \quad y_i = f(x_i)$$

  - Strict interpolation is in general not possible
    - In polynomials: n+1 points define a unique interpolation polynomial of degree n.
    - So, if we have 1000 points and want a cubic polynomial, we probably won't find it…

Solving Linear Systems in LS Sense

- We have an over-determined linear system $n \times k$:

$$f(x_1) = \lambda_1 f_1(x_1) + \lambda_2 f_2(x_1) + \ldots + \lambda_k f_k(x_1) = y_1$$

$$f(x_2) = \lambda_1 f_1(x_2) + \lambda_2 f_2(x_2) + \ldots + \lambda_k f_k(x_2) = y_2$$

...

...

...

$$f(x_n) = \lambda_1 f_1(x_n) + \lambda_2 f_2(x_n) + \ldots + \lambda_k f_k(x_n) = y_n$$

Solving Linear Systems in LS Sense

- In matrix form:

$$\begin{pmatrix} f_1(x_1) & f_2(x_1) & \dots & f_k(x_1) \\ f_1(x_2) & f_2(x_2) & \dots & f_k(x_2) \\ & & \dots & \\ \vdots & \vdots & & \vdots \\ f_1(x_n) & f_2(x_n) & \dots & f_k(x_n) \end{pmatrix} \begin{pmatrix} \lambda_1 \\ \lambda_2 \\ \vdots \\ \lambda_k \end{pmatrix} = \begin{pmatrix} y_1 \\ y_2 \\ \vdots \\ y_n \end{pmatrix}$$

Solving Linear Systems in LS Sense

- In matrix form:

$$A\mathbf{v} = \mathbf{y}$$

where $A = \left(f_j(x_i) \right)_{i.j}$ is a rectangular $n \times k$ matrix, n > k

$$\mathbf{v} = (\lambda_1, \lambda_2, ..., \lambda_k)^T$$

$$\mathbf{y} = (y_1, y_2, ..., y_n)^T$$

Solving Linear Systems in LS Sense

- More constraints than variables – no exact solutions generally exist
- We want to find something that is an "approximate solution":

$$\tilde{\mathbf{v}} = \arg\min_{\mathbf{v}} \left\| A\mathbf{v} - \mathbf{y} \right\|^2$$

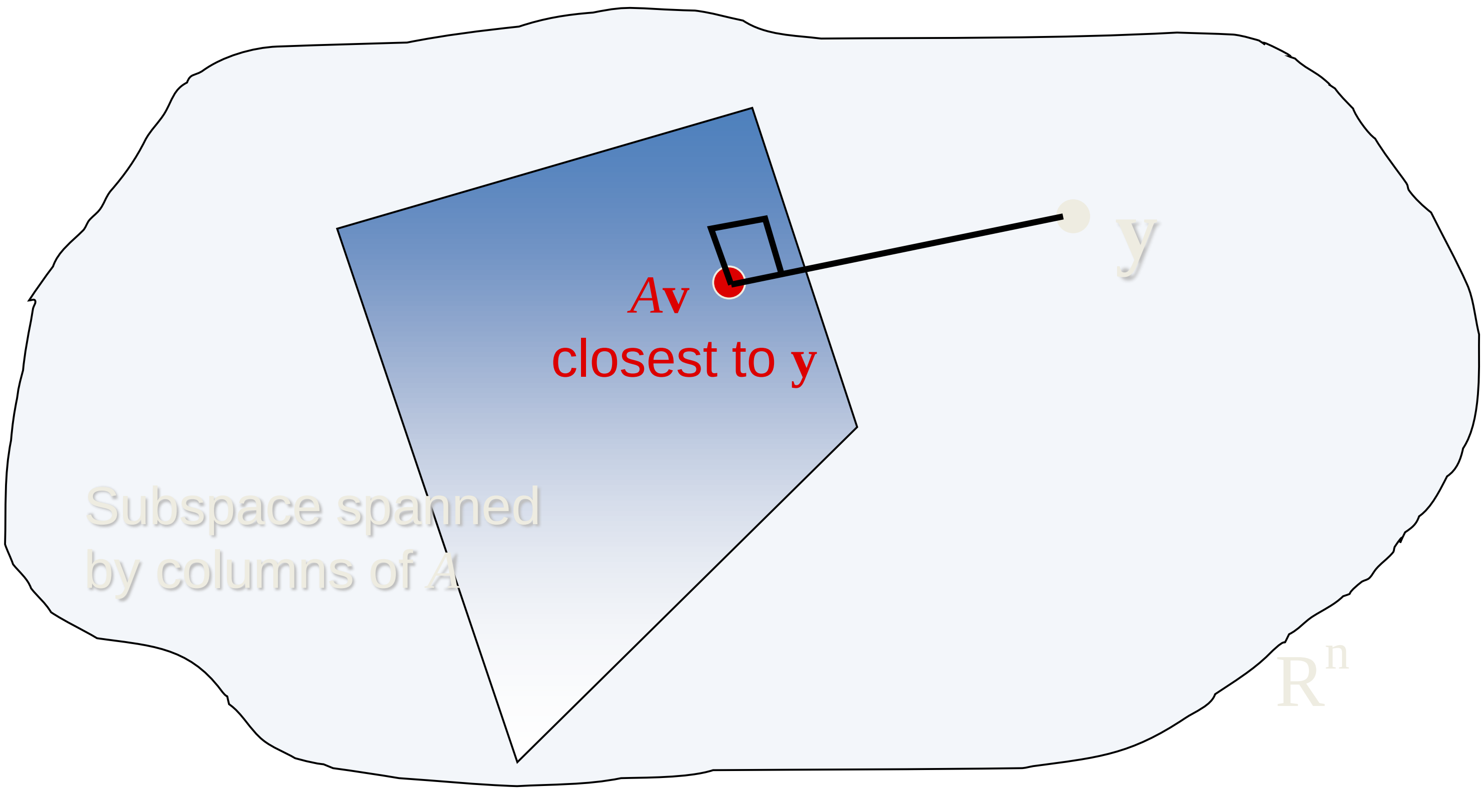
Finding the LS Solution

- $\mathbf{v} \in \mathrm{R}^k$
- $A\mathbf{v} \in \mathrm{R}^n$
- As we vary $\mathbf{v}$, $A\mathbf{v}$ varies over the linear subspace of R^n spanned by the columns of A:

$$A\mathbf{v} = \begin{pmatrix} A_1 & A_2 & \cdots & A_k \end{pmatrix} \begin{pmatrix} \lambda_1 \\ \lambda_2 \\ \cdot \\ \cdot \\ \lambda_k \end{pmatrix} = \lambda_1 A_1 + \lambda_2 A_2 + \ldots + \lambda_k A_k$$

Finding the LS Solution

- We want to find the closest $A\mathbf{v}$ to $\mathbf{y}$: $\min_{\mathbf{v}} \|A\mathbf{v} - \mathbf{y}\|^2$



Finding the LS Solution

- The vector $A\mathbf{v}$ closest to $\mathbf{y}$ satisfies:

$$(A\mathbf{v} - \mathbf{y}) \perp \{\text{subspace of } A\text{'s columns}\}$$

$$\Updownarrow$$

$$\forall \text{ column } A_i,\ \langle A_i, A\mathbf{v} - \mathbf{y} \rangle = 0$$

$$\forall\, i,\ A_i^{T}(A\mathbf{v} - \mathbf{y}) = 0$$

$$\Updownarrow$$

$$A^{T}(A\mathbf{v} - \mathbf{y}) = 0$$

$$(A^{T}A)\mathbf{v} = A^{T}\mathbf{y}$$

These are called the normal equations

Finding the LS Solution

- We got a square symmetric system $(A^{\mathrm{T}}A)\mathbf{v} = A^{\mathrm{T}}\mathbf{y}$

  (k×k)

- If A has full rank (the columns of A are linearly independent) then $(A^{\mathrm{T}}A)$ is invertible.

$$\min_{\mathbf{v}} \|A\mathbf{v} - \mathbf{y}\|^2$$

$$\Downarrow$$

$$\mathbf{v} = (A^T A)^{-1} A^T \mathbf{y}$$

Weighted Least Squares

- Sometimes the problem also has weights to the constraints:

$$\min_{\lambda_1,\lambda_2,\ldots,\lambda_k} \sum_{i=1}^{n} w_i [y_i - f_{\lambda_1,\lambda_2,\ldots,\lambda_k}(x_i)]^2, \; w_i > 0 \text{ and doesn't depend on } \lambda_{\text{i}}$$

$$\Updownarrow$$

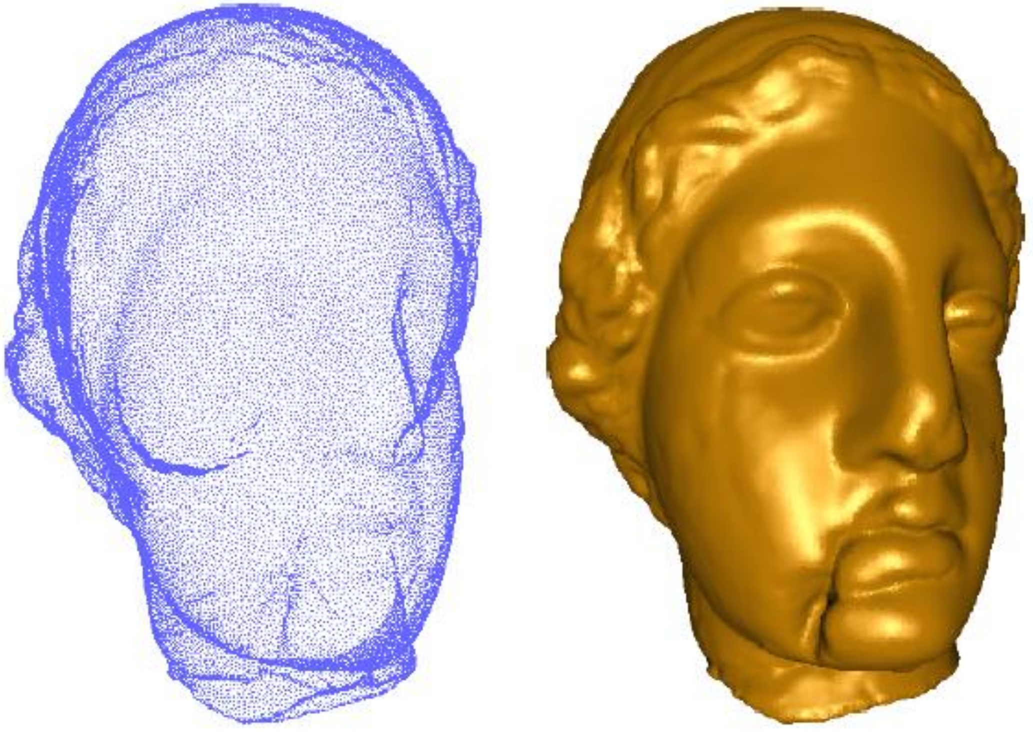
$$\min_{v} \; (A\mathbf{v} - \mathbf{y})^T W (A\mathbf{v} - \mathbf{y}), \;\; \text{where } \mathrm{W}_{\mathrm{ii}} = w_i \text{ is a diagonal matrix}$$

$$\Updownarrow$$

$$(A^T W A)v = A^T W y \quad \text{this is a square system}$$

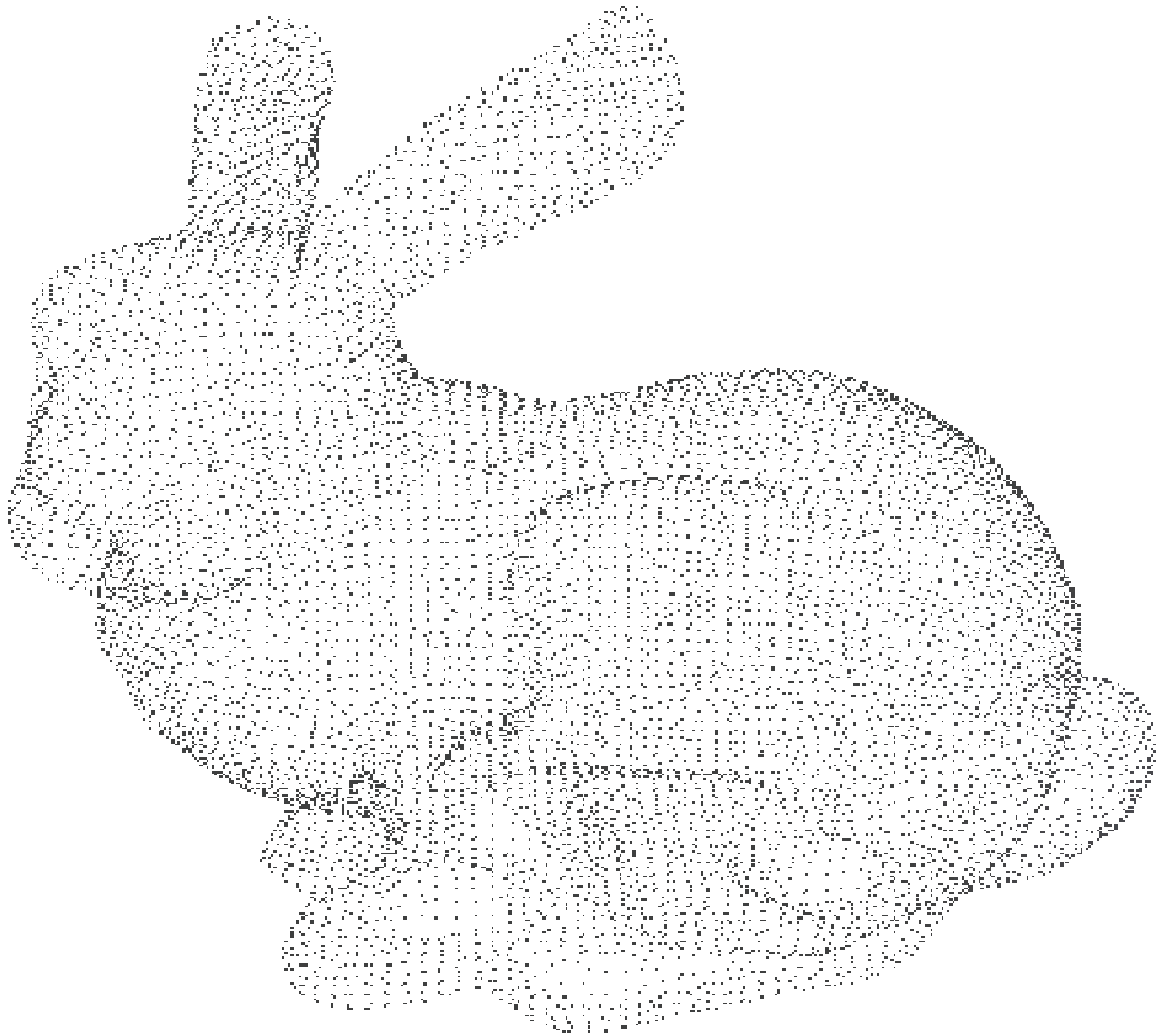
Motivation

- We are acquiring point cloud directly from scanners
- From physical prototypes to digital prototypes Local surface fitting to 3D points (Reverse Engineering



Local Surface Fitting to 3D points

- Normals?
- Lighting?
- Upsampling?

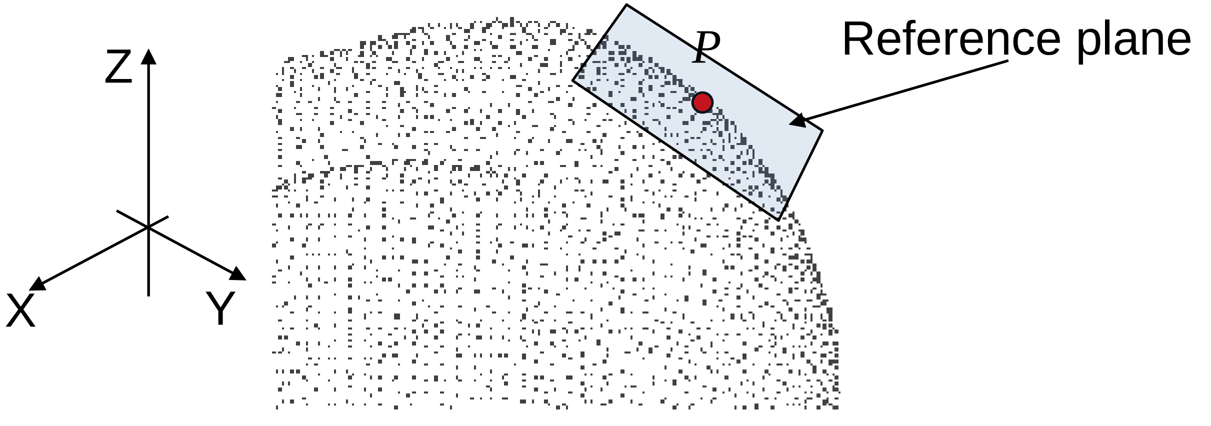


Local Surface Fitting to 3D points

Locally approximate
a polynomial surface
from points

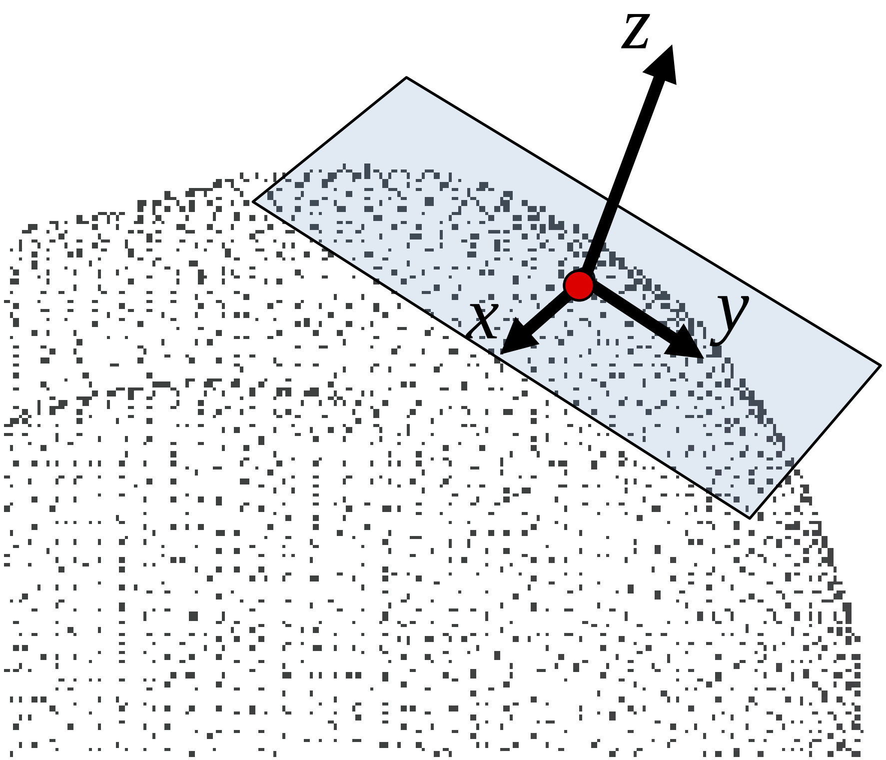
Fitting Local Polynomial

- Fit a local polynomial around a point P



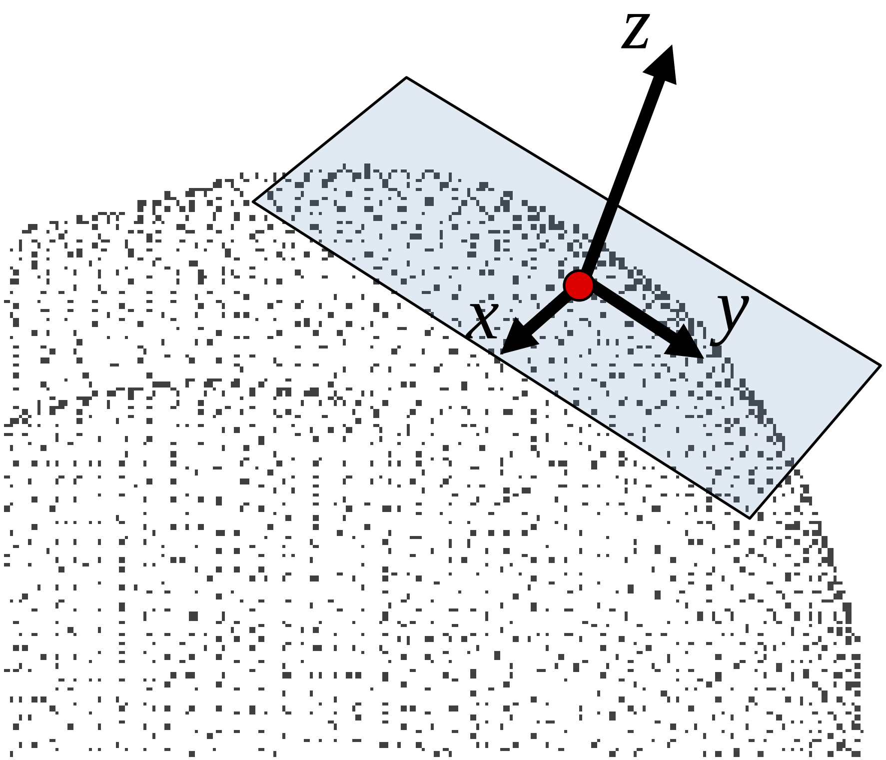
Fitting Local Polynomial Surface

- Compute a reference plane that fits the points close to P
- Use the local basis defined by the normal to the plane!



Fitting Local Polynomial Surface

- Fit polynomial $z = p(x,y) = ax^2 + bxy + cy^2 + dx + ey + f$



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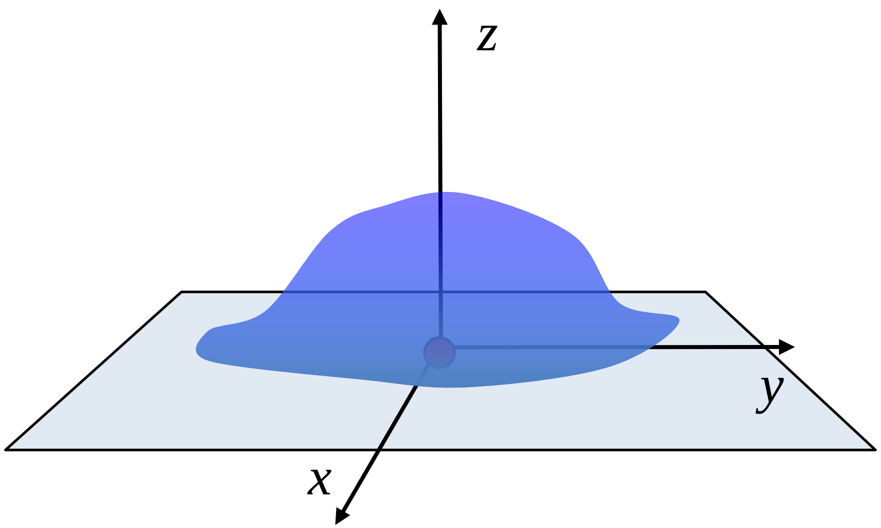

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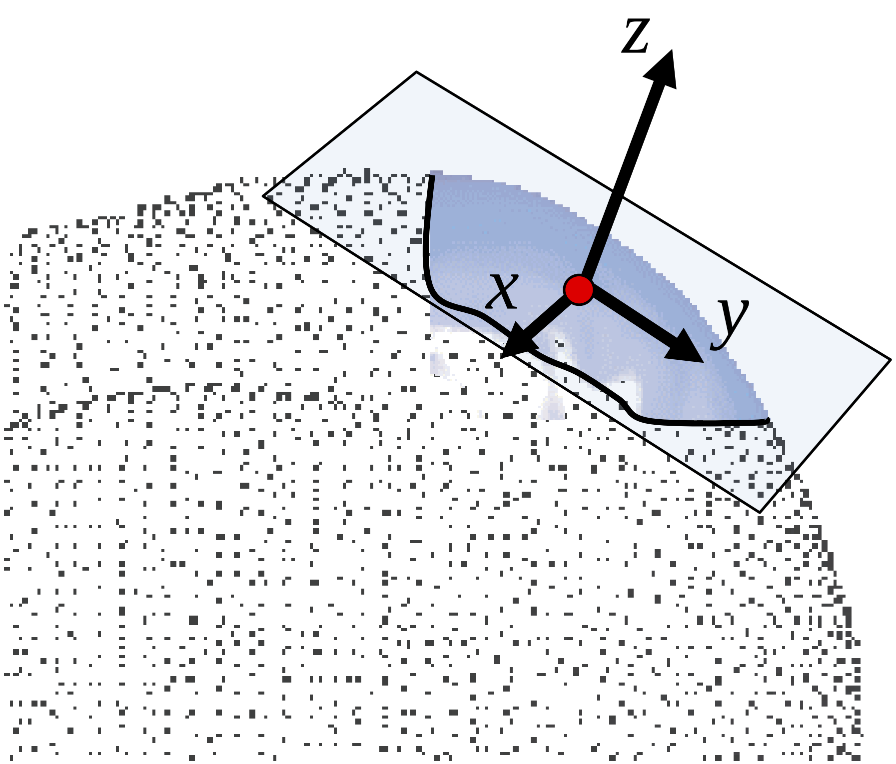

Fitting Local Polynomial Surface

- Again, solve the system in LS sense:

$$ax_1^2 + bx_1y_1 + cy_1^2 + dx_1 + ey_1 + f = z_1$$
$$ax_2^2 + bx_2y_2 + cy_2^2 + dx_2 + ey_2 + f = z_1$$
$$\cdots$$
$$ax_n^2 + bx_ny_n + cy_n^2 + dx_n + ey_n + f = z_n$$

- Minimize $\Sigma \|z_i - p(x_i, y_i)\|^2$

Fitting Local Polynomial Surface

- Also possible (and better) to add weights:

$$\Sigma\, w_i \,\|z_i - p(x_i, y_i)\|^2, \quad w_i > 0$$

- The weights get smaller as the distance from the origin point grows.