Shears

- **Shear operation**

$$\mathbf{S}h(z,a,b) = \begin{bmatrix} 1 & 0 & a & 0 \\ 0 & 1 & b & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

- **How about shears w.r.t. x-axis and y-axis**

Example

- **3D transformations are non-commutative in general**
- **Consider**

  (1) $\mathbf{A} = \mathbf{T}(2, 3, 0)$

  (2) $\mathbf{B} = \mathbf{R}(z, -90^0)$

$$\mathbf{A} \star \mathbf{B} = \begin{bmatrix} 1 & 0 & 0 & 2 \\ 0 & 1 & 0 & 3 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 0 & 1 & 0 & 0 \\ -1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

$$= \begin{bmatrix} 0 & 1 & 0 & 2 \\ -1 & 0 & 0 & 3 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

$$\mathbf{B} \star \mathbf{A} = \begin{bmatrix} 0 & 1 & 0 & 0 \\ -1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 & 2 \\ 0 & 1 & 0 & 3 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

$$= \begin{bmatrix} 0 & 1 & 0 & 3 \\ -1 & 0 & 0 & -2 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

- **Note that,**

$$\mathbf{A} \star \mathbf{B} \neq \mathbf{B} \star \mathbf{A}$$

Transformation Inverse

- Translation

$$\mathbf{T}^{-1}(\delta x, \delta y, \delta z) = \mathbf{T}(-\delta x, -\delta y, -\delta z)$$

- Rotation

$$\mathbf{R}^{-1}(x, \theta) = \mathbf{R}(x, -\theta)$$

$$\mathbf{R}^{-1}(y, \theta) = \mathbf{R}(y, -\theta)$$

$$\mathbf{R}^{-1}(z, \theta) = \mathbf{R}(z, -\theta)$$

- Scaling

$$\mathbf{S}^{-1}(a, b, c) = \mathbf{S}(\frac{1}{a}, \frac{1}{b}, \frac{1}{c})$$

- More complicated examples

$$(\mathbf{T}(\delta x, \delta y, \delta z)\mathbf{R}(z, -90^0))^{-1}$$

$$= \mathbf{R}^{-1}(z, -90^0)\mathbf{T}^{-1}(\delta x, \delta y, \delta z)$$

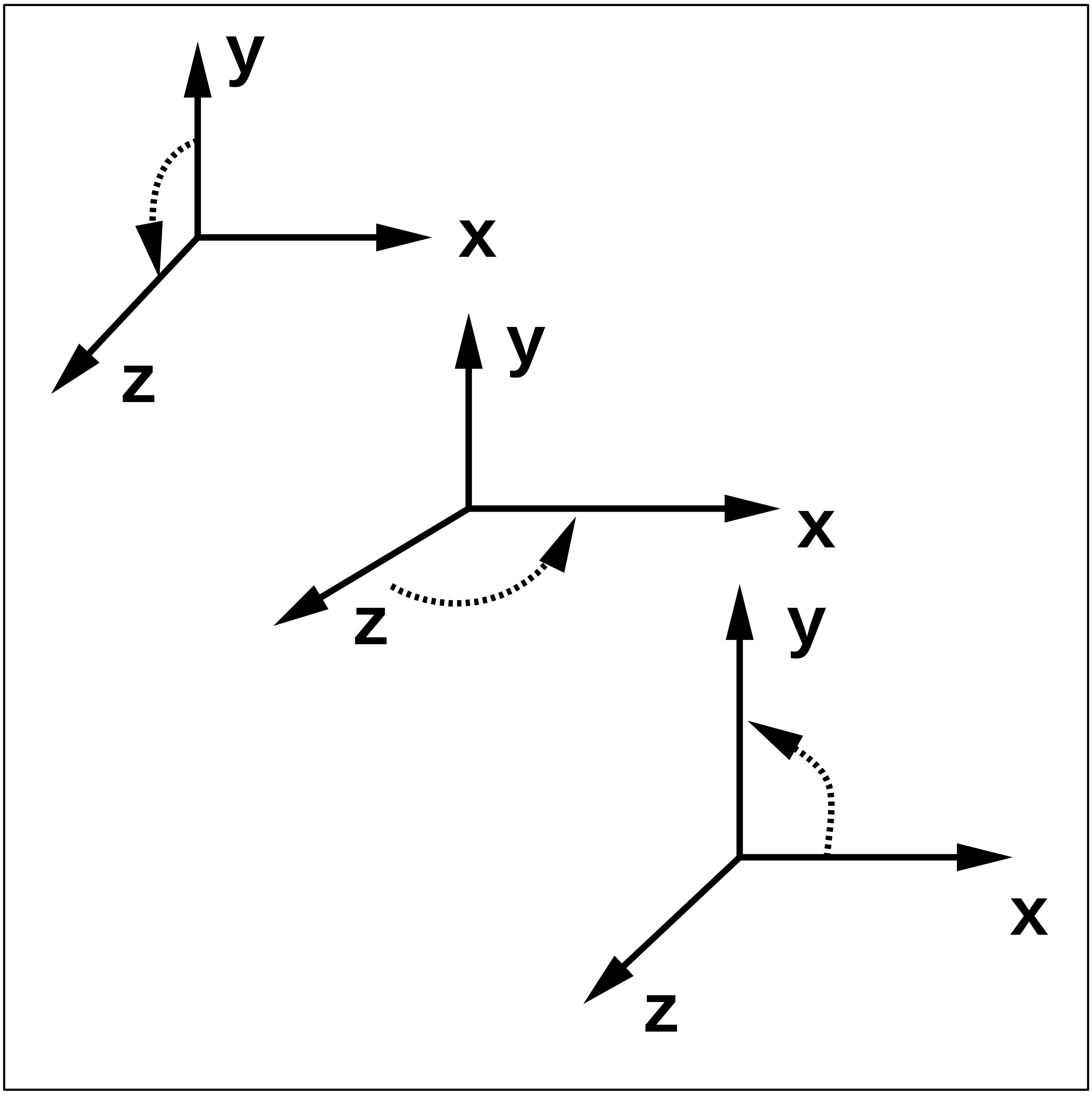
$$= \mathbf{R}(z, 90^0)\mathbf{T}(-\delta x, -\delta y, -\delta z)$$

$$(\mathbf{T}(a,b,c)\mathbf{R}(z,\alpha)\mathbf{R}(y,\beta))^{-1}$$

$$= \mathbf{R}^{-1}(y,\beta)\mathbf{R}^{-1}(z,\alpha)\mathbf{T}^{-1}(a,b,c)$$

$$= \mathbf{R}(y,-\beta)\mathbf{R}(z,-\alpha)\mathbf{T}(-a,-b,-c)$$

Positive Rotation



General Rotation

- Rotation about an arbitrary axis $\mathrm{R}(\mathrm{d}, \theta)$
- The axis is defined by a vector d

$$\mathbf{d} = \mathbf{b} - \mathbf{a}$$

- Translate d to the origin
  (now, d becomes d')

$$\mathbf{T}(-\mathbf{a}_x, -\mathbf{a}_y, -\mathbf{a}_z)$$

- Rotate about x-axis to bring d' to stay on x-z plane
  (now, d' becomes d'')

$$\mathrm{R}(x, \alpha)$$

How to determine α ?
α is determined by looking at projection
on the y-z plane
α needs not to be actually calculated,
only $\sin(\theta)$ and $\cos(\theta)$ matter,
they can be evaluated directly!

- **Rotate about y-axis to align d'' with z-axis (now, d'' becomes d''')**

$$\mathrm{R}(y, \beta)$$

Again, we do not actually need to compute β!

- **Perform the desired rotation**

$$\mathrm{R}(z, \theta)$$

- **Reverse all other steps**

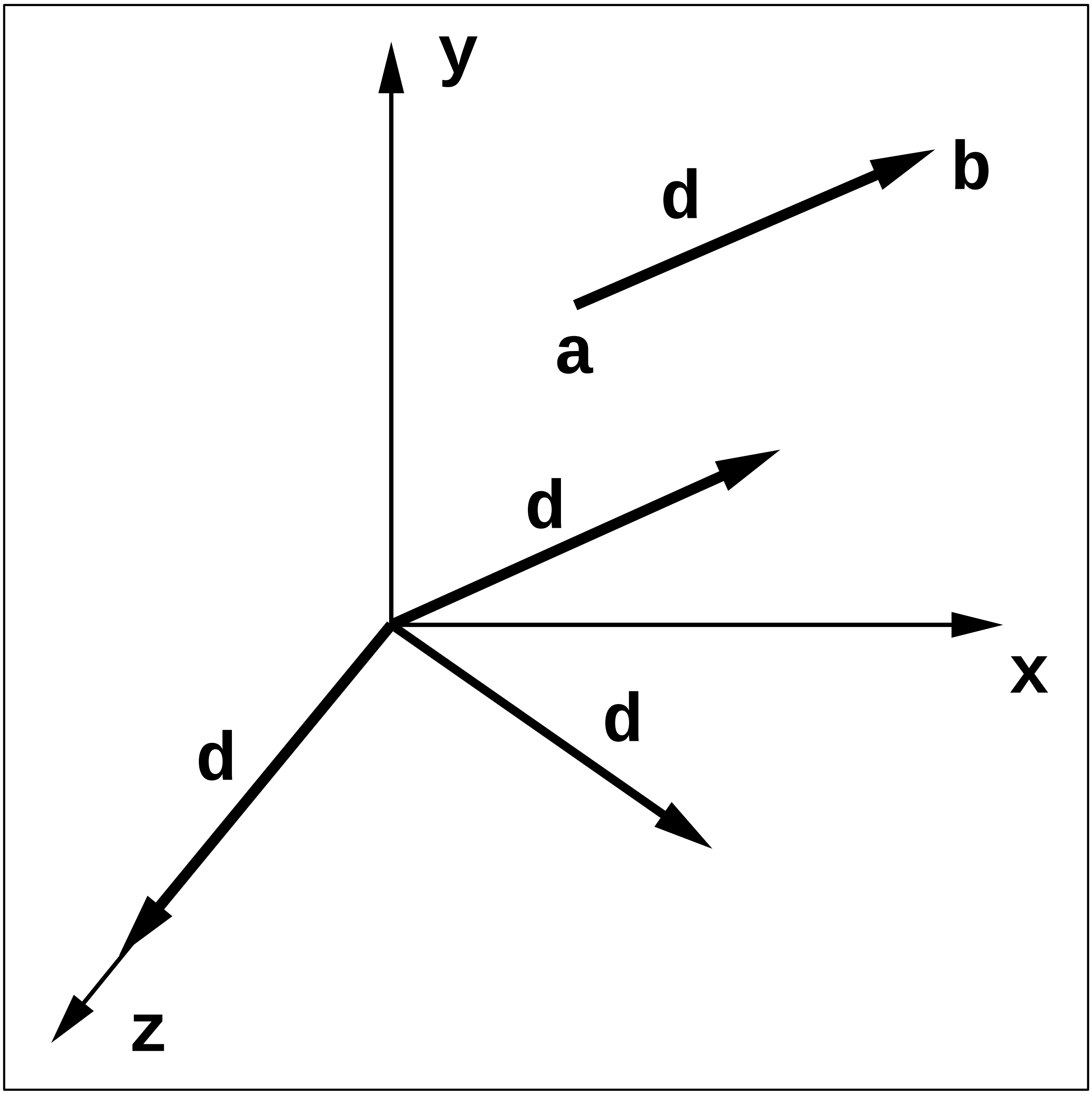
- **Overall**
  (1) $\mathrm{T}(-\mathrm{a})$
  (2) $\mathrm{R}(x, \alpha)$
  (3) $\mathrm{R}(y, \beta)$
  (4) $\mathrm{R}(z, \theta)$
  (5) $\mathrm{R}(y, -\beta)$
  (6) $\mathrm{R}(x, -\alpha)$
  (7) $\mathrm{T}(\mathrm{a})$

- **Let's put them together**

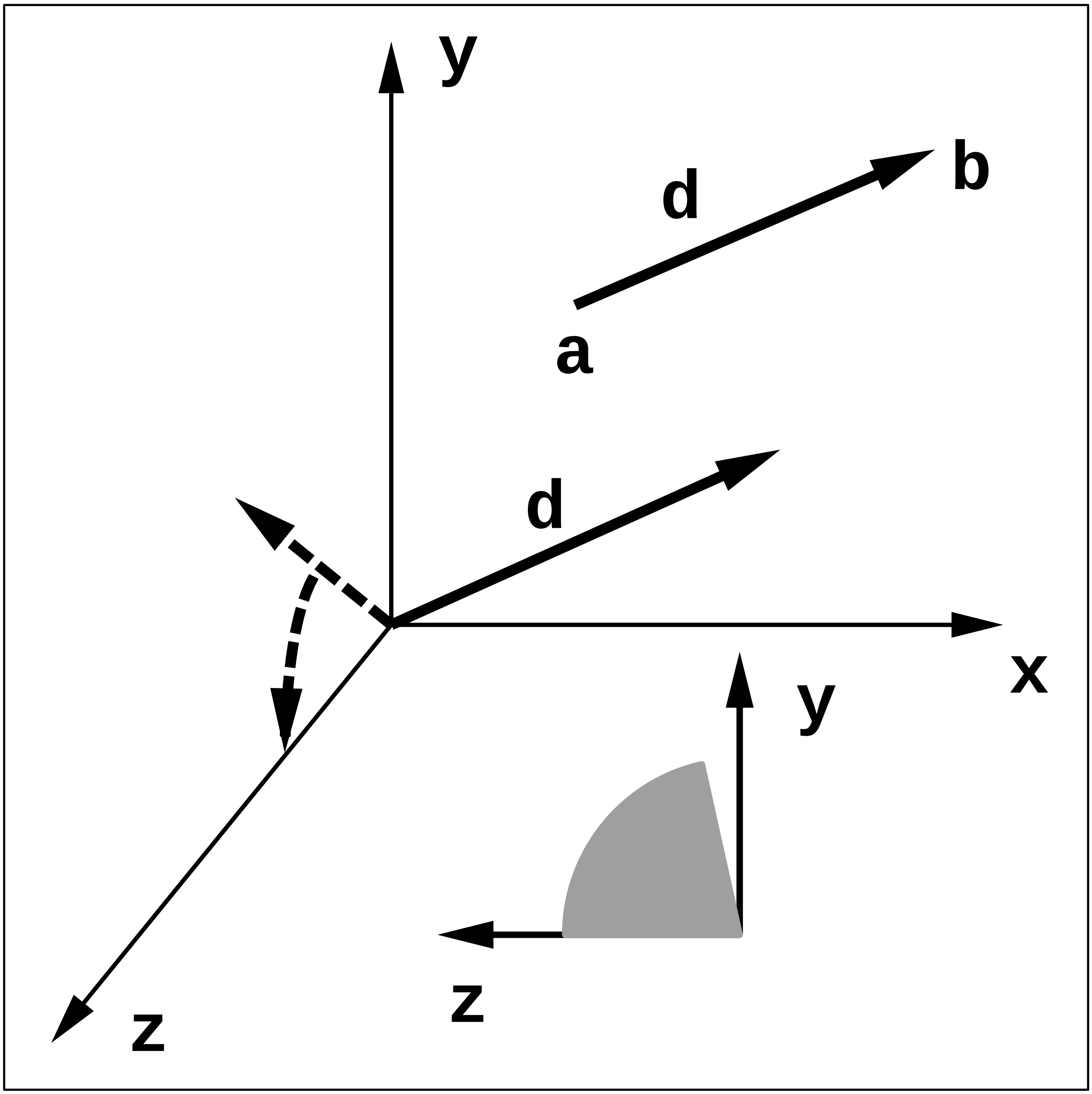
$$\mathrm{R}(\mathrm{d}, \theta) =$$

$$\mathrm{T}(\mathrm{a})\mathrm{R}(x, -\alpha)\mathrm{R}(y, -\beta)\mathrm{R}(z, \theta)(y, \beta)\mathrm{R}(x, \alpha)\mathrm{T}(-\mathrm{a})$$

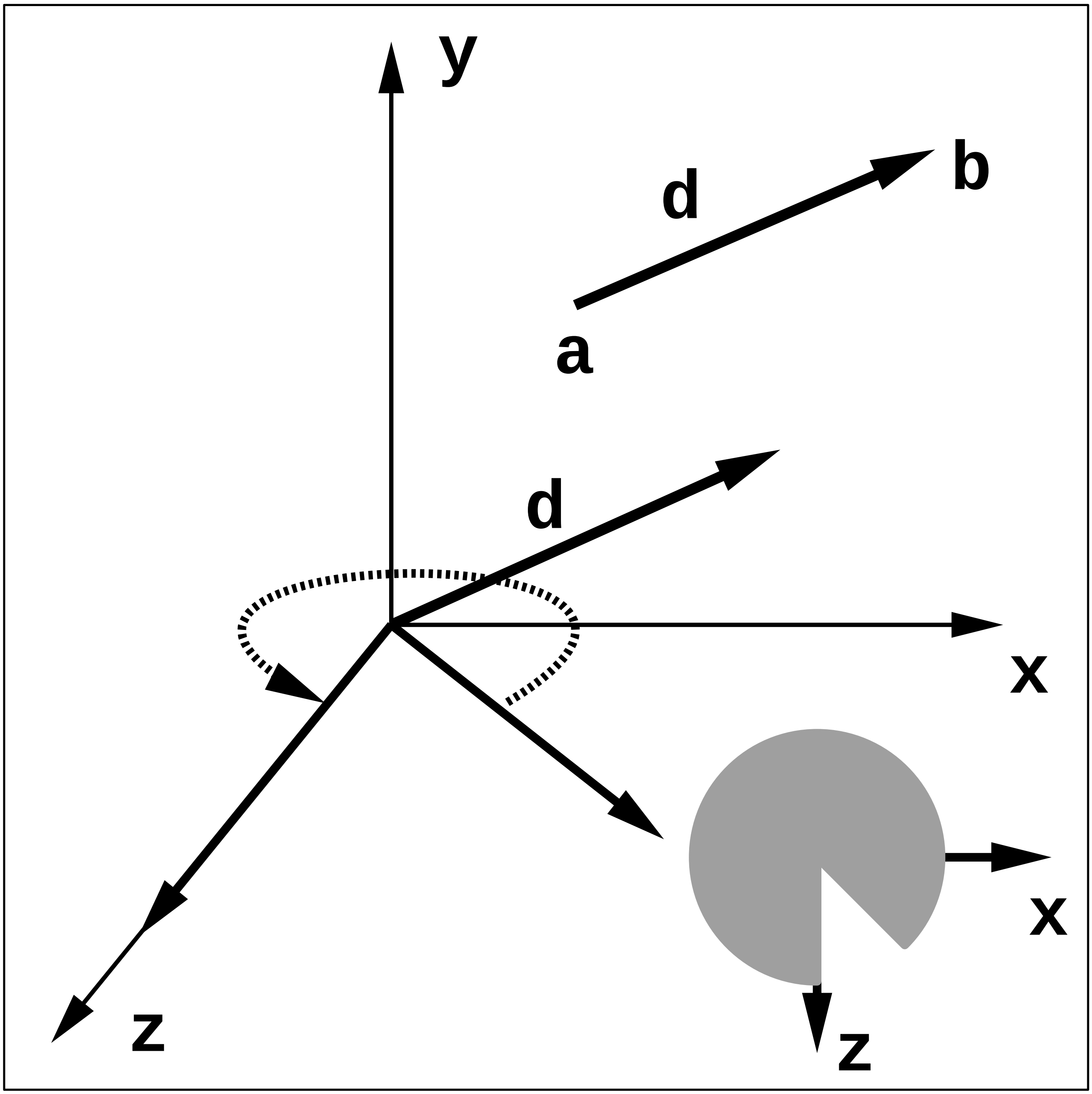
Arbitrary Rotation



Arbitrary Rotation



Arbitrary Rotation



Coordinate Systems

- **Transformation among coordinate systems**
- **Transformation can be thought of as a change in coordinate system**
- **How can we determine (different) coordinate values of the (same) object in (different) coordinate systems**
- **Consider point** $\mathbf{p}$

$$\mathbf{p}(x_1, y_1, z_1)$$

$$\mathbf{p}(x_2, y_2, z_2)$$

- **In CS-2**

$$\mathbf{p}(x_2, y_2, z_2) = x_2\mathbf{l} + y_2\mathbf{m} + z_2\mathbf{n}$$

- **In CS-1**

$$\mathbf{p}(x_1, y_1, z_1) = x_1\mathbf{i} + y_1\mathbf{j} + z_1\mathbf{k}$$

- **Connection**

$$\mathbf{p}(x_1, y_1, z_1) = \begin{bmatrix} \mathbf{t}_x \\ \mathbf{t}_y \\ \mathbf{t}_z \end{bmatrix} + x_2\mathbf{l} + y_2\mathbf{m} + z_2\mathbf{n}$$

Consider

$$\mathbf{l} = \begin{bmatrix} \mathbf{l}_x \\ \mathbf{l}_y \\ \mathbf{l}_z \end{bmatrix}$$

$$\mathbf{m} = \begin{bmatrix} \mathbf{m}_x \\ \mathbf{m}_y \\ \mathbf{m}_z \end{bmatrix}$$

$$\mathbf{n} = \begin{bmatrix} \mathbf{n}_x \\ \mathbf{n}_y \\ \mathbf{n}_z \end{bmatrix}$$

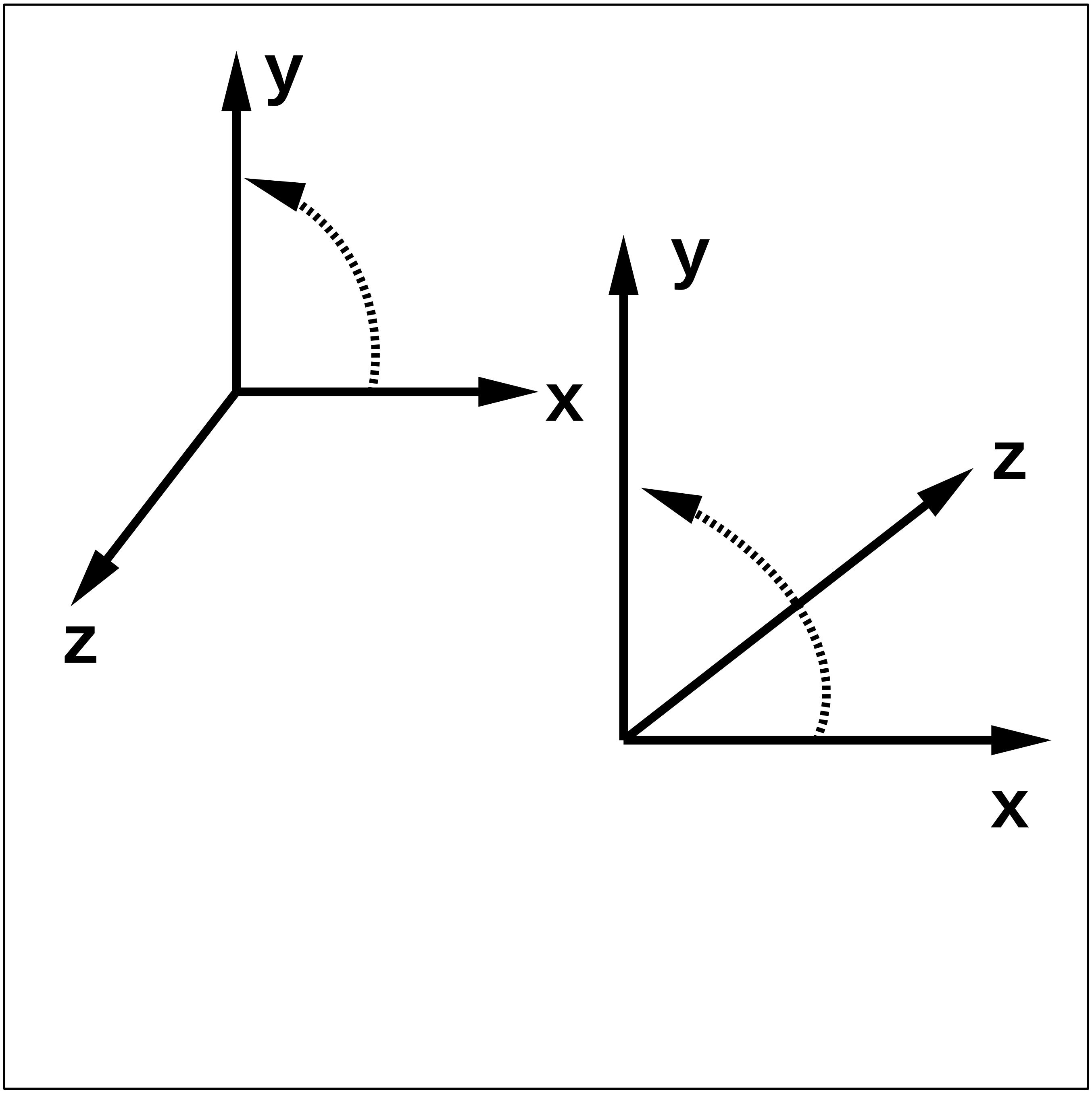
- **Let's put them together**

$$\begin{bmatrix} x_1 \\ y_1 \\ z_1 \\ 1 \end{bmatrix} = \begin{bmatrix} \mathbf{l}_x & \mathbf{m}_x & \mathbf{n}_x & \mathbf{t}_x \\ \mathbf{l}_y & \mathbf{m}_y & \mathbf{n}_y & \mathbf{t}_y \\ \mathbf{l}_z & \mathbf{m}_z & \mathbf{n}_z & \mathbf{t}_z \\ 0 & 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} x_2 \\ y_2 \\ z_2 \\ 1 \end{bmatrix}$$

- **Basis vectors of the local coordinate system**

expressed in the coordinates of the new (global) coordinate system

RH vs. LH



RH vs. LH

- **Conversion to left-handed system**

$$\begin{bmatrix} x_l \\ y_l \\ z_l \\ 1 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} x_r \\ y_r \\ z_r \\ 1 \end{bmatrix}$$