1 YES/NO questions

1. For any binary relation \( R \subseteq A \times A \), \( R^* \) exists.
   Justify: definition

2. \( R^* = R \cup \{(a, b) : \text{there is a path from } a \text{ to } b\} \).
   Justify: book definition

3. \( R^* = R \) for \( R = \{(a, b), (b, c), (a, c)\} \).
   Justify: \((a, a) \in R^* \) (trivial path from \( a \) to \( a \) always exist) but \((a, a) \notin R\)

4. All infinite sets have the same cardinality.
   Justify: \(|N| < |2^N|\) by Cantor Theorem and \( N, 2^N \) are infinite

5. Set \( A \) is uncountable iff \( R \subseteq A \) (\( R \) is the set of real numbers).
   Justify: \( R, 2^R \) are both uncountable and \( R \) is not a subset of \( 2^R \) (\( R \not\subseteq 2^R \)) but \( R \in 2^R \).

6. Let \( A \neq \emptyset \) such that there are exactly 25 partitions of \( A \). It is possible to define 20 equivalence relations on \( A \).
   Justify: one can define up to 25 (as many as partitions) of equivalence classes

7. There is a relation that is equivalence and order at the same time.
   Justify: equality relation

8. Let \( A = \{n \in N : n^2 + 1 \leq 15\} \). It is possible to define 8 alphabets \( \Sigma \subseteq A \).
   Justify: \( A \) has 4 elements, so we have \( 2^4 > 8 \) subsets

9. There is exactly as many languages over alphabet \( \Sigma = \{a\} \) as real numbers.
   Justify: \( |\Sigma^*| = \aleph_0, |2^{\Sigma^*}| = |R| = C. \)

10. Let \( \Sigma = \{a, b\} \). There are more than 20 words of length 4 over \( \Sigma \).
    Justify: There are exactly \( 2^4 = 16 \) words of length 4 over \( \Sigma \) and 16 < 20.

11. \( L^* = \{w_1...w_n : w_i \in L, i = 1, 2, ..n, n \geq 1\} \).
    Justify: \( n \geq 0 \).
    \( L^+ = LL^* \).
    Justify: the problem is only with cases \( e \in L \) or \( e \notin L \). When \( e \in L \), then \( e \in L^+ \), and always \( e \in L^* \), hence \( e \in LL^* \).
    When \( e \notin L \), then \( e \notin L^+ \), and always \( e \in L^* \), hence \( e \in LL^* \) and \( L^+ \neq LL^* \)

12. \( L^+ = L^* - \{e\} \).
    Justify: only when \( e \notin L \). When \( e \in L \) we get that \( e \in L^+ \) and \( e \notin L^* - \{e\} \).
13. If $L = \{ w \in \{0, 1\}^* : w \text{ has an unequal number of 0's and 1's} \}$, then $L^* = \{0, 1\}^*$.

**Justify:** $1 \in L, 0 \in L$ so $\{0, 1\} \subseteq L \subseteq \Sigma^*$, hence $\{0, 1\}^* \subseteq L^* \subseteq (\Sigma^*)^* = \Sigma^* = \{0, 1\}^*$ and $L^* = \{0, 1\}^*$.

14. For any languages $L_1, L_2$, $(L_1 \cup L_2) \cap L_1 = L_1$.

**Justify:** languages are sets and $(A \cup B) \cap A = A$.

15. For any languages $L_1, L_2$,

$$L_1^* = L_2^* \text{ iff } L_1 = L_2$$

**Justify:** Consider $L_1 = \{a, e\}, L_2 = \{a\}$. Obviously, $L_1 \neq L_2$ and $L_1^* = L_2^*$.

16. For any languages $L_1, L_2$, $(L_1 \cup L_2)^* = L_1^*$.

**Justify:** languages are sets so it is true only when $L_1 \subseteq L_2$.

17. $((\emptyset \cap a) \cup b^*) \cap \emptyset^*$ describes a language with only one element.

**Justify:** $\emptyset \cup b^* = b^*$, $b^* \cap \{e\} = \{e\}$

18. $((\emptyset \cap a) \cup b^*) \cap a^*$ is a finite regular language.

**Justify:** $b^* \cap a^* = \{e\} = \emptyset^*$

19. $\{\{a\} \cup \{e\}\} \cap \{ab\}^*$ is a finite regular language.

**Justify:** $\{\{a\} \cup \{e\}\} \cap \{ab\}^* = \{a, e\} \cap \{ab\}^* = \{e\} = \emptyset^*$

20. Any regular language has a finite description.

**Justify:** by definition $L = \mathcal{L}(r)$ and $r$ is a finite string.

21. Any finite language is regular.

**Justify:** $L = \{w_1\} \cup \ldots \cup \{w_1\}$ and $\{w_1\}$ has a finite description $\{w_1\}$

22. Every deterministic automata is also non-deterministic.

**Justify:** any function is a relation

The set of all configurations of any non-deterministic state automata is always non-empty.

**Justify:** $K \neq \emptyset$, because $s \in K$. If $\Sigma = \emptyset$ the set of all configuration of non-deterministic automata (book definition) is a subset of $K \times \emptyset \cup \{e\} \neq \emptyset$ as it always contains $(s, e)$. For the lecture definition, the set of all configuration is a subset of $K \times \Sigma^*$ and always $e \in \Sigma^*$ hence always $(s, e) \in K \times \Sigma^*$.

23. Let $M$ be a finite state automaton, $L(M) = \{ w \in \Sigma^* : (q, w) \xrightarrow{\Sigma, M} (s, e) \}$.

**Justify:** $L(M) = \{ w \in \Sigma^* : \exists q \in F((s, w) \xrightarrow{\Sigma, M} (q, e)) \}$

24. For any automata $M$, $L(M) \neq \emptyset$.

**Justify:** if $\Sigma = \emptyset$ or $F = \emptyset$, $L(M) = \emptyset$

25. $L(M_1) = L(M_2)$ iff $M_1, M_2$ are deterministic.

**Justify:** Let $M_1$ be an automata over $\{a, b\}$ with with $\Delta = \{ (q_0, ab, q_0) \}$, $F = \{ q_0 \}$, $s = q_0$ and let $M_2$ be an automata over $\{a, b\}$ with with $\Delta = \{ (q_0, ab, q_0), (q_0, e, q_1) \}$, $F = \{ q_1 \}$, $s = q_0$.

$L(M_1) = L(M_2) = (ab)^*$ and both are non-deterministic
26. DFA and NDFA compute the same class of languages.
   **Justify:** basic theorem

27. Let $M_1$ be a deterministic, $M_2$ be a nondeterministic FA, $L_1 = L(M_1)$ and $L_2 = L(M_2)$ then there is a deterministic automaton $M$ such that $L(M) = (L_1^* \cup (L_1 - L_2)^*)L_1$
   **Justify:** the class of finite automata is closed under $\ast, \cup, -, \cap$

**TWO DEFINITIONS OF NON DETERMINISTIC AUTOMATA**

**BOOK DEFINITION:** $M = (K, \Sigma, \Delta, s, F)$ is non-deterministic when
$$\Delta \subseteq K \times (\Sigma \cup \{e\}) \times K$$

OBSERVE that $\Delta$ is always finite because $K, \Sigma$ are finite sets.

**LECTURE DEFINITION:** $M = (K, \Sigma, \Delta, s, F)$ is non-deterministic when $\Delta$ is finite and
$$\Delta \subseteq K \times \Sigma^* \times K$$

OBSERVE that we have to say in this case that $\Delta$ is finite because $\Sigma^*$ is infinite.

**SOLVING PROBLEMS** you can use any of these definitions.

## 2 Problems

**Problem 1** Let $L$ be a language defines as follows
$$L = \{w \in \{a, b\}^* : \text{every a is either immediately proceeded or followed by b}\}.$$

1. Describe a regular expression $r$ such that $L(r) = L$ (Meaning of $r$ is $L$).
   **Solution** $L = (b \cup ab \cup ba \cup bab)^*$

2. Construct a finite state automata $M$, such that $L(M) = L$.
   **Solution**
   Components of $M$ are:
   - $K = \{s\}, \{a, b\}, s, F = \{s\},$
   - $\Delta = \{(s, b, s), (s, ab, s), (s, ba, s), (s, bab, s)\}.$
   Some elements of $L(M)$ are: $b, bb, baab, abab, abbbbba, bbbabbbabbabb$

**Problem 2**

1. Let $M = (K, \Sigma, \delta, s, F)$ be a deterministic finite automaton. Under exactly what conditions $e \in L(M)$?
   **Solution**
   $$e \in L(M) \iff s \in F.$$

2. Let $M = (K, \Sigma, \Delta, s, F)$ be a non-deterministic finite automaton. Under exactly what conditions $e \in L(M)$?
Solution  Now we have two possibilities: $s \in F$ (computation of length 0) or there is a computation of length $> 0$ from $(s, e)$ to $(q, e)$ for $q \in F$ when $s \notin F$.

Problem 3  Let 

$$M = (K, \Sigma, s, \Delta, F)$$

for $K = \{q_0, q_1, q_2, q_3\}$, $s = q_0$

$\Sigma = \{a, b\}$, $F = \{q_1, q_2, q_3\}$ and

$$\Delta = \{(q_0, a, q_1), (q_0, b, q_3), (q_1, a, q_2), (q_1, b, q_1), (q_3, a, q_3), (q_3, b, q_2)\}$$

1. List some elements of $L(M)$.

Solution  $a, b, aa, bb, aba, abba$

2. Write a regular expression for the language accepted by $M$. Simplify the solution.

Solution  

$$L(M) = ab^* \cup ab^* a \cup ba^* \cup ba^* b = ab^*(e \cup a) \cup ba^*(e \cup b).$$

3. Define a deterministic $M'$ such that $M \approx M'$, i.e. $L(M) = L(M')$.

Solution  We complete $M$ do a deterministic $M'$ by adding a TRAP state $q_4$ and put

$$\Delta' = \delta = \Delta \cup \{(q_2, a, q_4), (q_2, b, q_4), (q_4, a, q_4), (q_4, b, q_4)\}$$

Justify  why $M \approx M'$.

Solution  $q_4$ is a trap state, it does not influence $L(M)$.

Problem 4  Let $M$ be defined as follows

$$M = (K, \Sigma, s, \Delta, F)$$

for $K = \{q_0, q_1, q_2, q_3\}$, $s = q_0$

$\Sigma = \{a, b, c\}$, $F = \{q_0, q_2, q_3\}$ and

$$\Delta = \{(q_0, abc, q_0), (q_0, a, q_1), (q_0, c, q_3), (q_1, bc, q_1), (q_1, b, q_2), (q_2, a, q_2), (q_2, b, q_3), (q_3, a, q_3)\}$.

Find the regular expression describing the $L(M)$. Simplify it as much as you can. Explain your steps. Does $e \in L(M)$?

Solution  

$$L = (abc)^*a(bc)^* e \cup a^* ba^*.$$

Observe that $e \in L$ as $q_0 \in F$ and also $(q_0, e, q_3) \in \Delta$ and $q_3 \in F$.

Write down (you can draw the diagram) an automata $M'$ such that $M' \equiv M$ and $M'$ is defined by the BOOK definition.

Solution  

Solution  We apply the "stretching" technique to $M$ and the new $M'$ is as follows.

$$M' = (K \cup \{p_1, p_2, p_3\}, \Sigma, s = q_0, \Delta', F' = F)$$

for $K = \{q_0, q_1, q_2\}$, $s = q_0$

$\Sigma = \{a, b\}$, $F = \{q_0, q_2, q_3\}$ and

$$\Delta' = \{(q_0, a, q_1), (q_0, e, q_3), (q_1, b, q_2), (q_2, a, q_2), (q_2, b, q_3), (q_3, a, q_3)\} \cup \{(q_0, a, p_1), (p_1, b, p_2), (p_2, c, q_0), (q_1, b, p_3), (p_3, b, q_1)\}.$$
Problem 5 For $M$ defined as follows

$$M = (K, \Sigma, s, \Delta, F)$$

for $K = \{q_0, q_1, q_2, q_3\}$, $s = q_0$

$\Sigma = \{a, b\}$, $F = \{q_2\}$ and

$$\Delta = \{(q_0, a, q_3), (q_0, c, q_3), (q_0, b, q_1), (q_0, c, q_1), (q_1, a, q_2), (q_2, b, q_3), (q_2, c, q_3)\}$$

Write 2 steps of the general method of transformation the NDFA $M$ defined above into an equivalent DFA $M'$.

Step 1: Evaluate $\delta(E(q_0), a)$ and $\delta(E(q_0), b)$.

Step 2: Evaluate $\delta$ on all states that result from step 1.

Reminder: $E(q) = \{p \in K : (q, e) \xrightarrow{\delta} (p, e)\}$ and

$$\delta(Q, \sigma) = \bigcup\{E(p) : \exists q \in Q, (q, \sigma, p) \in \Delta\}.$$

Solution Step 1: First we need to evaluate $E(q)$, for all $q \in K$.

$$E(q_0) = \{q_0, q_1, q_3\} = S, E(q_1) = \{q_1\}, E(q_2) = \{q_2, q_3\} \in F, E(q_3) = \{q_3\}$$

$$\delta(E(q_0), a) = \delta(\{q_0, q_1, q_3\}, a) = E(q_3) \cup E(q_2) \cup \emptyset = \{q_2, q_3\} \in F$$

$$\delta(E(q_0), b) = \delta(\{q_0, q_1, q_3\}, b) = E(q_1) \cup \emptyset \cup \emptyset = \{q_1\}$$

Solution Step 2:

$$\delta(\{q_2, q_3\}, a) = \emptyset \cup \emptyset = \emptyset$$

$$\delta(\{q_2, q_3\}, b) = E(q_3) \cup \emptyset = \{q_3\}$$

$$\delta(\{q_1\}, a) = E(q_2) = \{q_2, q_3\} \in F$$

$$\delta(\{q_1\}, b) = \emptyset$$