1 YES/NO questions

1. For any function \( f \) from \( A \neq \emptyset \) onto \( A \), \( f \) has property \( f(a) \neq a \) for certain \( a \in A \).
   \textbf{Justify:} \( f(x) = x \) is always "onto".
   \textbf{n}

2. All infinite sets have the same cardinality.
   \textbf{Justify:} \( |N| \neq |R| \) and \( N \) (natural numbers) and \( R \) (real numbers) are infinite sets.
   \textbf{n}

3. \( \{a, b\} \in 2^{\{a, b, \{a, b\}\}} \)
   \textbf{Justify:} \( \{a, b\} \subseteq \{a, b, \{a, b\}\} \).
   \textbf{y}

4. For any binary relation \( R \subseteq A \times A \), \( R^{-1} \) exists.
   \textbf{Justify:} The set \( R^{-1} = \{(b, a): (a, b) \in R\} \) always exists.
   \textbf{y}

5. Regular language is a regular expression.
   \textbf{Justify:} Regular language is a language defined by a regular expression.
   \textbf{n}

6. \( L^+ = \{w_1...w_n: w_i \in L, i = 1, 2, ..n, n \geq 1\} \)
   \textbf{Justify:} definition
   \textbf{y}

7. \( L^+ = L^* - \{e\} \).
   \textbf{Justify:} only when \( e \notin L \). When \( e \in L \) we get that \( e \in L^+ \) and \( e \notin L^* - \{e\} \).
   \textbf{n}

8. For any languages \( L_1, L_2, (L_1 \cap L_2) \cup L_2 = L_2 \).
   \textbf{Justify:} \( L_1 \cap L_2 \subseteq L_2 \) and languages are sets.
   \textbf{y}

9. \( (\emptyset^* \cap b^*) \cup \emptyset^* \) describes a language with only one element.
   \textbf{Justify:} \( (\{e\} \cap \{b\}^*) \cup \{e\} = \{e\} \)
   \textbf{y}

10. For any \( M, L(M) \neq \emptyset \) iff the set \( F \) of its final states is non-empty.
    \textbf{Justify:} Let \( M \) be such that \( \Sigma = \emptyset, F \neq \emptyset, s \notin F \), we get \( L(M) = \emptyset \).
     \textbf{n}

11. A configuration of any finite automaton \( M = (K, \Sigma, \Delta, s, F) \) is any element of \( K \times \Sigma^* \times K \).
    \textbf{Justify:} it is element of \( K \times \Sigma^* \)
    \textbf{n}

12. If \( M = (K, \Sigma, \Delta, s, F) \) is a non-deterministic as defined in the book, then \( M \) is also non-deterministic, as defined in the lecture.
    \textbf{Justify:} \( \Sigma \cup \{e\} \subseteq \Sigma^* \)
    \textbf{y}

13. Let \( M \) be a finite state automaton, \( L(M) = \{\omega \in \Sigma^*: (s, \omega) \xrightarrow{s, M} (q, e)\} \).
    \textbf{Justify:} only when \( q \in F \)
    \textbf{n}

14. \( L(M_1) = L(M_2) \) iff \( M_1, M_2 \) are finite automata.
    \textbf{Justify:} one can have 2 automata that accept different languages.
    \textbf{n}

15. DFA and NDFA recognize the same class of languages.
    \textbf{Justify:} theorem proved in class
    \textbf{y}
Two definitions of a non-deterministic automaton

**BOOK DEFINITION:** \( M = (K, \Sigma, \Delta, s, F) \) is non-deterministic when
\[ \Delta \subseteq K \times (\Sigma \cup \{e\}) \times K \]

**OBSERVE** that \( \Delta \) is always finite because \( K, \Sigma \) are finite sets.

**LECTURE DEFINITION:** \( M = (K, \Sigma, \Delta, s, F) \) is non-deterministic when \( \Delta \) is finite and
\[ \Delta \subseteq K \times \Sigma^* \times K \]

**OBSERVE** that we have to say in this case that \( \Delta \) is finite because \( \Sigma^* \) is infinite.

**SOLVING PROBLEMS** you can use any of these definitions.

3 Very short questions (25pts)

For all state diagrams below do the following.

1. Determine whether it defines a finite state automaton.
2. Determine whether it is a deterministic / non-deterministic automaton.
3. Write full definition of \( M \) by listing all its components.
4. Describe the language by writing a regular expression or a property that defines it.

**Q1 Solution:** \( M = (K, \Sigma, s, \Delta, F) \) for \( K = \{ q_0 \} = F, s = q_0, \Sigma = \emptyset, \Delta = \emptyset \). \( M \) is deterministic and
\[ L(M) = \{ e \} \neq \emptyset \]

**Q2 Solution:** \( M = (K, \Sigma, s, \Delta, F) \) for \( \Sigma = \{ a, b \}, K = \{ q_0, q_1 \}, s = q_0, F = \{ q_0 \}, \Delta = \{ (q_0, a, q_1), (q_1, b, q_0) \} \). \( M \) is non deterministic; \( \Delta \) is not a function on \( K \times \Sigma \).
\[ L(M) = (ab)^* \]

**Q3 Solution:** \( M = (K, \Sigma, s, \Delta, F) \) for \( \Sigma = \{ a, b \}, K = \{ q_0, q_1, q_2, q_3 \}, s = q_0, F = \emptyset, \Delta = \{ (q_0, a, q_1), (q_0, b, q_1), (q_1, a, q_1), (q_1, e, q_2), (q_2, ab, q_2) \} \). It is NOT an automaton. It has no initial state.

**Q4** \( M = (K, \Sigma, s, \Delta, F) \) for \( \Sigma = \{ a, b \}, K = \{ q_0, q_1, q_2, q_3 \}, s = q_0, F = \emptyset, \Delta = \{ (q_0, a, q_1), (q_1, b, q_2), (q_2, a, q_1), (q_0, e, q_3), (q_2, a, q_3) \} \). \( M \) is non deterministic; \( \Delta \subseteq K \times \Sigma \cup \{e\} \times K \).
\[ L(M) = \emptyset \]

**Q5** \( M = (K, \Sigma, s, \Delta, F) \) for \( \Sigma = \{ a, b \}, K = \{ q_0, q_1, q_2, q_3 \}, s = q_0, F = \{ q_1 \}, \Delta = \{ (q_0, ab, q_1), (q_1, e, q_0), (q_1, a, q_2), (q_1, ba, q_2), (q_2, a, q_2), (q_0, e, q_3), (q_1, a, q_3) \} \). \( M \) is non deterministic; \( \Delta \subseteq K \times \Sigma^* \times K, q_2, q_3 \) are trap states.
\[ L(M) = (ab)^+ \]
4 Problems

Problem 1 Let \( L \) be a language defines as follows
\[ L = \{ w \in \{a, b\}^* : \text{between any two a's in w there is an even number of consequitive b's.} \} \]

1. Describe a regular expression \( r \) such that \( L(r) = L \).

Solution Remark that 0 is an even number, hence \( a^* \in L \),
\[ r = b^* \cup b^*ab^* \cup b^*(a(bb)^*a)^*b^* = b^*ab^* \cup b* (a(bb)^*a)^*b^* \]

2. Construct a finite state automata \( M \), such that \( L(M) = L \).

Solution 1 Components of \( M \) are:
\[ \Sigma = \{a, b\}, K = \{q_0, q_1, q_2, q_3\}, s = q_0, F = \{q_0, q_2, q_3\} ,\]
\[ \Delta = \{(q_0, b, q_0), (q_0, a, q_3), (q_0, a, q_1), (q_1, bb, q_1), (q_1, a, q_2), (q_2, e, q_0), (q_3, b, q_3)\} \]

Some elements of \( L(M) \) as defined by the state diagram are:
\[ a, aaa, bbb, aaaaabb, bbbaaa, abba, ababbbba, abbbbbabba, .... \]

Solution 2 Components of \( M \) are:
\[ \Sigma = \{a, b\}, K = \{q_0, q_1, q_2\}, s = q_0, F = \{q_0, q_1, q_2\} ,\]
\[ \Delta = \{(q_0, b, q_0), (q_0, e, q_1), (q_1, bb, q_1), (q_1, a, q_1), (q_1, c, q_2), (q_2, b, q_2)\} \]

Problem 2 Let \( M = (K, \Sigma, s, \Delta, F) \)
for \( K = \{q_0\}, s = q_0, \Sigma = \{a, b\}, F = \{q_0\} \) and
\[ \Delta = \{(q_0, aba, q_0), (q_0, ab, q_0)\} \]

1. List some elements of \( L(M) \).

Solution
\[ e, ab, abab, ababa, ababaaba, ... \]

2. Write a regular expression for the language accepted by \( M \).

Solution
\[ L = (ab \cup aba)^* \]
Problem 3 We know that for any deterministic finite automaton \( M = (K, \Sigma, s, \delta, F) \) the following is true:
\[
e \in L(M) \text{ iff } s \in F.
\]
Show that the above is not true for all non-deterministic automata.

Solution Let \( M = (K, \Sigma, s, \Delta, F) \) for \( K = \{q_0, q_1\}, s = q_0, \Sigma = \emptyset, F = \{q_1\}, \) and \( \Delta = \{(q_0, e, q_1)\} \).
\[
L(M) = \{e\} \text{ and } s \notin F.
\]

Problem 4 For \( M \) defined as follows
\[
M = (K, \Sigma, s, \Delta, F)
\]
for \( K = \{q_0, q_1, q_2, q_3\}, s = q_0 \)
\[
\Sigma = \{a, b\}, F = \{q_2, q_3\} \text{ and } \\
\Delta = \{(q_0, a, q_1), (q_0, e, q_1), (q_0, b, q_2), (q_1, b, q_3), (q_1, e, q_3), (q_2, b, q_2), (q_2, e, q_3), (q_3, a, q_3)\}
\]
Write a regular expression describing \( L(M) \).

Solution Regular expression (not simplified) describing \( L(M) \) is:
\[
aa^* \cup a^* \cup aba^* \cup ba^* \cup bb^* \cup bb^*a^*
\]

Write 4 steps of the general method of transformation the NDFA \( M \), into an equivalent deterministic \( M' \).

Reminder: \( E(q) = \{p \in K : (q, e) \overset{M}{\rightarrow} (p, e)\} \) and
\[
\delta(Q, \sigma) = \bigcup \{E(p) : \exists q \in Q, (q, \sigma, p) \in \Delta\}.
\]

Solution Step 1:
\[
E(q_0) = \{q_0, q_1, q_3\}, \quad E(q_1) = \{q_1, q_3\}, \quad E(q_2) = \{q_2, q_3\}, \quad E(q_3) = \{q_3\}.
\]

Solution Step 2:
\[
\delta(E(q_0), a) = \delta(\{q_0, q_1, q_3\}, a) = E(q_1) \cup E(q_3) = \{q_1, q_3\} \in F,
\]
\[
\delta(E(q_0), b) = \delta(\{q_0, q_1, q_3\}, b) = E(q_2) \cup E(q_3) \cup \emptyset = \{q_2, q_3\} \in F.
\]

Solution Step 3:
\[
\delta(\{q_1, q_3\}, a) = \emptyset \cup E(q_3) = \{q_3\} \in F,
\]
\[
\delta(\{q_1, q_3\}, b) = E(q_3) \cup \emptyset = \{q_3\} \in F,
\]
\[
\delta(\{q_2, q_3\}, a) = \emptyset \cup E(q_3) = \{q_3\} \in F,
\]
\[
\delta(\{q_2, q_3\}, b) = E(q_2) = \{q_2, q_3\}
\]

Solution Step 4:
\[
\delta(\{q_3\}, a) = E(q_3) = \{q_3\} \in F, \quad \delta(\{q_3\}, b) = \emptyset,
\]
\[
\delta(\emptyset, a) = \emptyset, \quad \delta(\emptyset, b) = \emptyset.
\]

End of the construction.