1 YES/NO questions

1. For any function \( f \) from \( A \neq \emptyset \) onto \( A \), \( f \) has property \( f(a) \neq a \) for certain \( a \in A \).
   Justify: \( f(x) = x \) is always "onto".

2. All infinite sets have the same cardinality.
   Justify: \( |N| \neq |R| \) and \( N \) (natural numbers) and \( R \) (real numbers) are infinite sets.

3. \( \{\{a,b\}\} \in 2^{\{a,b,\{a,b\}\}} \)
   Justify: \( \{\{a,b\}\} \subseteq \{a,b,\{a,b\}\} \).

4. For any binary relation \( R \subseteq A \times A \), \( R^{-1} \) exists.
   Justify: The set \( R^{-1} = \{(b,a) : (a,b) \in R\} \) always exists.

5. Regular language is a regular expression.
   Justify: Regular language is a language defined by a regular expression.

6. \( L^+ = \{w_1...w_n : w_i \in L, i = 1, 2, ..n, n \geq 1\} \)
   Justify: definition

7. \( L^* = L^+ \cup \emptyset \)
   Justify: only when \( e \not\in L \). When \( e \in L \) we get that \( e \in L^+ \) and \( e \notin L^* \).

8. For any languages \( L_1, L_2, (L_1 \cup L_2) \cup \emptyset \neq L_2 \).
   Justify: \( L_1 \cup \emptyset \subseteq L_2 \) and languages are sets.

9. \( (\emptyset \cup b^*) \cup \emptyset^* \) describes a language with only one element.
   Justify: \( (\emptyset \cup \{b\}) \cup \{e\} = \{e\} \)

10. For any \( M, L(M) \neq \emptyset \) iff the set \( F \) of its final states is non-empty.
    Justify: Let \( M \) be such that \( \Sigma = \emptyset, F \neq \emptyset, s \notin F \), we get \( L(M) = \emptyset \).

11. A configuration of any finite automaton \( M = (K, \Sigma, \Delta, s, F) \) is any element of \( K \times \Sigma^* \times K \).
    Justify: it is element of \( K \times \Sigma^* \) (lecture definition)

12. If \( M = (K, \Sigma, \Delta, s, F) \) is a non-deterministic as defined in the book, then \( M \) is also non-deterministic, as defined in the lecture.
    Justify: \( \Sigma \cup \{e\} \subseteq \Sigma^* \)

13. Let \( M \) be a finite state automaton, \( L(M) = \{\omega \in \Sigma^* : (s, \omega) \xrightarrow{s,M} (q,e)\} \).
    Justify: only when \( q \in F \)

14. \( L(M_1) = L(M_2) \) iff \( M_1, M_2 \) are finite automata.
    Justify: one can have 2 automata that accept different languages.

15. DFA and NDFA recognize the same class of languages.
    Justify: theorem proved in class
2 Two definitions of a non-deterministic automaton

**BOOK DEFINITION:** $M = (K, \Sigma, \Delta, s, F)$ is non-deterministic when

$$\Delta \subseteq K \times (\Sigma \cup \{e\}) \times K$$

**OBSERVE** that $\Delta$ is always finite because $K, \Sigma$ are finite sets.

**LECTURE DEFINITION:** $M = (K, \Sigma, \Delta, s, F)$ is non-deterministic when $\Delta$ is finite and

$$\Delta \subseteq K \times \Sigma^* \times K$$

**OBSERVE** that we have to say in this case that $\Delta$ is finite because $\Sigma^*$ is infinite.

**SOLVING PROBLEMS** you can use any of these definitions.

3 Very short questions (25pts)

For all state diagrams below do the following.

1. Determine whether it defines a finite state automaton.
2. Determine whether it is a deterministic / non-deterministic automaton.
3. Write full definition of $M$ by listing all its components.
4. Describe the language by writing a regular expression or a property that defines it.

**Q1 Solution:** $M = (K, \Sigma, s, \Delta, F)$ for $K = \{q_0\} = F$, $s = q_0$, $\Sigma = \emptyset$, $\Delta = \emptyset$. $M$ is deterministic and

$L(M) = \{e\} \neq \emptyset$

**Q2 Solution:** $M = (K, \Sigma, s, \Delta, F)$ for $\Sigma = \{a, b\}$, $K = \{q_0, q_1\}$, $s = q_0$, $F = \{q_0\}$, $\Delta = \{(q_0, a, q_1), (q_1, b, q_0)\}$. $M$ is non deterministic; $\Delta$ is not a function on $K \times \Sigma$.

$L(M) = (ab)^*$

**Q3 Solution:** $M = (K, \Sigma, s, \Delta, F)$ for $\Sigma = \{a, b\}$, $K = \{q_0, q_1, q_2, q_3\}$, $s = q_0$, $F = \emptyset$, $\Delta = \{(q_0, a, q_1), (q_1, b, q_2), (q_2, a, q_1), (q_0, e, q_3), (q_2, a, q_3)\}$. $M$ is non deterministic; $\Delta \subseteq K \times (\Sigma \cup \{e\}) \times K$.

$L(M) = \emptyset$

**Q4 Solution:** $M = (K, \Sigma, s, \Delta, F)$ for $\Sigma = \{a, b\}$, $K = \{q_0, q_1, q_2, q_3\}$, $s = q_0$, $F = \emptyset$, $\Delta = \{(q_0, a, q_1), (q_1, e, q_0), (q_1, a, q_2), (q_1, b, q_2), (q_2, a, q_2), (q_2, a, q_3)\}$. $M$ is non deterministic; $\Delta \subseteq K \times \Sigma^* \times K$, $q_2, q_3$ are trap states.

$L(M) = (ab)^+$
4 Problems

Problem 1 Let $L$ be a language defined as follows

$$L = \{w \in \{a, b\}^* : \text{between any two } a\text{'s in } w \text{ there is an even number of consecutive } b\text{'s.}\}.$$

1. Describe a regular expression $r$ such that $L(r) = L$.

Solution Remark that 0 is an even number, hence $a^* \in L$,

$$r = b^* \cup b^*ab^* \cup b^*(a(bb)^*a)^*b^* = b^*ab^* \cup b^*(a(bb)^*a)^*b^*$$

2. Construct a finite state automata $M$, such that $L(M) = L$.

Solution 1 Components of $M$ are:

$$\Sigma = \{a, b\}, K = \{q_0, q_1, q_2, q_3\}, s = q_0, F = \{q_0, q_2, q_3\}$$

$$\Delta = \{(q_0, b, q_0), (q_0, a, q_3), (q_0, a, q_1), (q_1, bb, q_1), (q_1, a, q_2), (q_2, e, q_0), (q_3, b, q_3)\}$$

Some elements of $L(M)$ as defined by the state diagram are:

$$a, aaa, bbb, aaaaabbb, bbbaaa, abba, abbabbbba, abbbbbbabba, \ldots.$$ 

Solution 2 Components of $M$ are:

$$\Sigma = \{a, b\}, K = \{q_0, q_1, q_2\}, s = q_0, F = \{q_0, q_1, q_2\}$$

$$\Delta = \{(q_0, b, q_0), (q_0, e, q_1), (q_1, bb, q_1), (q_1, a, q_1), (q_1, c, q_2), (q_2, b, q_2)\}$$

Problem 2 Let

$$M = (K, \Sigma, s, \Delta, F)$$

for $K = \{q_0\}, s = q_0, \Sigma = \{a, b\}, F = \{q_0\}$ and

$$\Delta = \{(q_0, aba, q_0), (q_0, ab, q_0)\}$$

1. List some elements of $L(M)$.

Solution

$$e, ab, abab, ababa, ababaaba, \ldots$$

2. Write a regular expression for the language accepted by $M$.

Solution

$$L = (ab \cup aba)^*$$
Problem 3  We know that for any deterministic finite automaton $M = (K, \Sigma, s, \delta, F)$ the following is true:

$$e \in L(M) \iff s \in F.$$  

Show that the above is not true for all non-deterministic automata.

Solution Let $M = (K, \Sigma, s, \Delta, F)$ for $K = \{q_0, q_1\}, s = q_0, \Sigma = \emptyset, F = \{q_1\},$ and $\Delta = \{(q_0, e, q_1)\}$. 

$L(M) = \{e\}$ and $s \notin F$.

Problem 4  For $M$ defined as follows

$$M = (K, \Sigma, s, \Delta, F)$$

for $K = \{q_0, q_1, q_2, q_3\}$, $s = q_0$

$\Sigma = \{a, b\}$, $F = \{q_2, q_3\}$ and

$$\Delta = \{(q_0, a, q_1), (q_0, e, q_1), (q_0, b, q_2), (q_1, b, q_3), (q_1, e, q_3), (q_2, b, q_2), (q_2, e, q_3), (q_3, a, q_3)\}$

Write a regular expression (not simplified) describing $L(M)$.

Solution  Regular expression (not simplified) describing $L(M)$ is:

$$aa^* \cup a^* \cup aba^* \cup ba^* \cup bb^* \cup ba^*$$

Write 4 steps of the general method of transformation the NDFA $M$, into an equivalent deterministic $M'$.

Reminder: $E(q) = \{p \in K : (q, e) \xrightarrow{M} (p, e)\}$ and

$$\delta(Q, \sigma) = \bigcup\{E(p) : \exists q \in Q, (q, \sigma, p) \in \Delta\}.$$

Solution Step 1:

$E(q_0) = \{q_0, q_1, q_3\}$, $E(q_1) = \{q_1, q_3\}$, $E(q_2) = \{q_2, q_3\}$, $E(q_3) = \{q_3\}$.

Solution Step 2:

$$\delta(E(q_0), a) = \delta(\{q_0, q_1, q_3\}, a) = E(q_1) \cup E(q_3) = \{q_1, q_3\} \in F,$$

$$\delta(E(q_0), b) = \delta(\{q_0, q_1, q_3\}, b) = E(q_2) \cup E(q_3) \cup \emptyset = \{q_2, q_3\} \in F,$$

Solution Step 3:

$$\delta(\{q_1, q_3\}, a) = \emptyset \cup E(q_3) = \{q_3\} \in F,$$

$$\delta(\{q_1, q_3\}, b) = E(q_3) \cup \emptyset = \{q_3\} \in F,$$

$$\delta(\{q_2, q_3\}, a) = \emptyset \cup E(q_3) = \{q_3\} \in F,$$

$$\delta(\{q_2, q_3\}, b) = E(q_3) = \{q_2, q_3\}$$

Solution Step 4:

$$\delta(\{q_3\}, a) = E(q_3) = \{q_3\} \in F, \quad \delta(\{q_3\}, b) = \emptyset,$$

$$\delta(\emptyset, a) = \emptyset, \quad \delta(\emptyset, b) = \emptyset.$$

End  of the construction.