cse303
ELEMENTS OF THE THEORY OF COMPUTATION

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LECTURE 4a
REVIEW FOR CHAPTER 1

1. Some Short Questions
2. Some Homework Problems
CHAPTER 1

SHORT QUESTIONS
Short YES/NO Questions

Here are solutions to some short YES/NO Questions for material covered in CHAPTER 1

Solving Quizzes and Tests you have to write a short solutions and circle the answer

You will get 0 pts if you only circle your answer without providing a solution, even if it is correct answer

Here are some questions
Q1 \( \{\emptyset, \{\emptyset\}\} \cap \{\emptyset\} \neq \emptyset \)

yes

We have that

\[ \emptyset \in \{\emptyset\} \quad \text{and} \quad \emptyset \in \{\emptyset, \{\emptyset\}\} \]

This proves that

\[ \emptyset \in \{\emptyset\} \cap \{\emptyset, \{\emptyset\}\} \]

Hence \( \{\emptyset, \{\emptyset\}\} \cap \{\emptyset\} \neq \emptyset \)
Short YES/NO Questions

Q2 Some relations \( R \subseteq A \times B \) are functions that map the set \( A \) into the set \( B \)

**yes**

Functions are special type of relations so some binary relations are functions (but not all relations are functions)

Q3 \( 2^\emptyset = \emptyset \)

**no**

\( \emptyset \subseteq \emptyset \) so \( \emptyset \in 2^\emptyset \)
Short YES/NO Questions

Q4  For any binary relation \( R \) on a set \( A \),
the inverse relation \( R^{-1} \) exists

yes

By definition of the inverse relation is

\[
R^{-1} = \{(b, a) : (a, b) \in R\}
\]

and such set always exists
Short YES/NO Questions

Q5 For any function $f : A \longrightarrow B$, the inverse function $f^{-1} : B \longrightarrow A$ exists

no

Inverse function to a function $f$ exists if and only if $f$ is 1–1 and onto

Q6 A set $A = \{x \in N : x^2 + 5 < 0\}$ is countable

yes

$A = \{x \in N : x^2 + 5 < 0\} = \emptyset$

and any finite set is countable
Q7 The set \( A = \{ n \in \mathbb{N} : n^2 + 5 > 0 \} \) is countable.

Yes

The definition says:
A set \( A \) is countable if and only if it is finite or is countably infinite.

The set
\[
A = \{ n \in \mathbb{N} : n^2 + 5 > 0 \} = \mathbb{N}
\]

and \( \mathbb{N} \) is countably infinite, hence \( A \) is countable.
The set \( A = \black\{(\{n\}, n) \in 2^N \times N : 1 \leq n \leq n^2\} \) is infinitely countable.

First observe that

\[
A = \black\{(\{n\}, n) \in 2^N \times N : 1 \leq n \leq n^2\} = C \times B
\]

where the set \( B \) is

\[
B = \{n \in N : 1 \leq n \leq n^2\}
\]

and the set \( C \) is

\[
C = \black\{\{n\} \in 2^N : 1 \leq n \leq n^2\}
\]
Short YES/NO Questions

The condition \( 1 \leq n \leq n^2 \) holds for all \( n \in N - \{0\} \) hence the set

\[
B = \{ n \in N : 1 \leq n \leq n^2 \}
\]

is infinitely countable and so is the set

\[
C = \{ \{n\} \in 2^N : 1 \leq n \leq n^2 \}
\]

as the function \( f(n) = \{n\} \) is \( 1 - 1 \) and maps \( B \) onto \( C \)

The set

\[
A = C \times B
\]

is infinitely countable as it is the cartesian product of two infinitely countable sets
Q9  
Let \( A = \{ n \in N : n^2 + 1 \leq 15 \} \)

It is possible to define **8 alphabets** \( \Sigma \subseteq A \)

The set

\[
A = \{ n \in N : n^2 + 1 \leq 15 \} = \{0, 1, 2, 3\}
\]

so the set \( A \) has 4 elements and it has \( 2^4 = 16 \) of all possible subsets and they are all finite, i.e we can define up to up to **16 alphabets** \( \Sigma \subseteq A \)

So have can define for sure **8 < 16** alphabets
Q10  Let $\Sigma = \{ n \in N : n^2 + 1 = 10 \}$
There are **uncountably** many **finite** languages over $\Sigma$

Observe that

$$\Sigma = \{ n \in N : n^2 + 1 = 10 \} = \{ 3 \}$$

and hence $|\Sigma^*| = \aleph_0$

A **finite** language over $\Sigma$ is by definition a **finite** subset of $\Sigma^*$

We have a Theorem:

The set of all **finite subsets** of any countably infinite set is **countably infinite**
Short YES/NO Questions

Q11  For any languages $L_1$, $L_2$, $L$ over $\Sigma \neq \emptyset$ we have that

$$(L_1 \cup L_2) \cap L = (L_1 \cap L) \cup (L_2 \cap L)$$

yes  Languages are sets hence all laws of algebra of sets hold for them and this is one of the Distributivity laws

Q12  $L^* = \{w_1w_2\ldots w_n : w_i \in L, i = 1, 2, \ldots n, n \geq 1\}$

no  This is the definition of $L^+$; we must put $n \geq 0$ for $L^*$
A regular language is a regular expression.

A regular language is represented by a regular expression.

More precisely, a regular language is represented by the function \( L : \text{Regular Expressions} \rightarrow \text{Regular Languages} \) such that the following holds:

if \( \alpha \) is any regular expression, then \( L(\alpha) \) is the language represented by \( \alpha \).
Short YES/NO Questions

Q14 Let $\alpha = a(a \cup b)^*$

$$\mathcal{L}(\alpha) = \{w \in \{a, b\}^* : w \text{ ends with } a\}$$

no

We evaluate

$$\mathcal{L}(a(a \cup b)^*) = \{a\}(\{a\} \cup \{b\})^* = \{a\}\Sigma^*$$

and hence the property defining $\mathcal{L}(\alpha)$ is

$$\mathcal{L}(\alpha) = \{w \in \{a, b\}^* : w \text{ starts with } a\}$$
Short YES/NO Questions

Q15 For any language $L$ over an alphabet $\Sigma$,

$$L^+ = L \cup L^*$$

no

Take $L$ be any language such that $e \notin L$

We have that

$$e \notin L^+ \text{ but } e \in L \cup L^*$$

This proves that

$$L^+ \neq L \cup L^*$$
CHAPTER 1

Some Homework Problems
Problem 1

Consider the following languages over $\Sigma = \{a, b\}$

$$L_1 = \{w \in \Sigma^* : \exists u \in \Sigma \Sigma (w = uu^R u)\}$$

$$L_2 = \{w \in \Sigma^* : ww = www\}$$

Part 1: Prove that $L_1$ is a finite set

Give example of 3 words $w \in L_1$

Solution

We evaluate first the set

$\Sigma \Sigma = \{a, b\}\{a, b\} = \{aa, bb, ab, ba\}$

$\Sigma \Sigma$ is a finite set, hence the set $B = \{uyu : u, y \in \Sigma \Sigma\}$ is also a finite set and by definition $L_1 \subseteq B$

This proves that $L_1$ must be a finite set
Problem 1

We evaluated that $\Sigma \Sigma = \{a, b\}\{a, b\} = \{aa, bb, ab, ba\}$

We defined $L_1 = \{w \in \Sigma^* : \exists u \in \Sigma \Sigma (w = uu^R u)\}$

By evaluation we have that

$$L_1 = \{aaaaaa, abbaab, baabba, bbbbbbb\}$$

Part 2: Give examples of 2 words over $\Sigma$ such that $w \notin L_1$

Solution $a \notin L_1$, $bba \notin L_1$

There are countably infinitely many words that are not in $L_1$
Problem 1

Part 3  Consider now the following language

\[ L_2 = \{ w \in \{a, b\}^* : \; ww = www \} \]

Show that \( L_2 \neq \emptyset \)

Solution  \( e \in L_2 \), as \( ee = eee \)

In fact, \( e \) is the only word with this property, hence

\[ L_2 = \{ e \} \]

Part 4  Show that the set \( (\Sigma^* - L_2) \) is infinite

Solution  \( \Sigma^* \) is countably infinite, \( L_2 \) is finite, so (basic theorem) \( (\Sigma^* - L_2) \) is countably infinite

Any \( w \in \Sigma^* \), such that \( w \neq e \) is in \( (\Sigma^* - L_2) \)
Problem 2

Part 1

Given expressions (written in a short hand notation)

\[ \alpha_1 = \emptyset^* \cup a^* \cup b^* \cup a \cup b \cup (a \cup b)^* \]

\[ \alpha_2 = (a \cup b)^* b(a \cup b)^* \]

Re-write \( \alpha_1 \) as a simpler expression representing the same language.

List properties you used in your solution.

Describe the language \( L = \mathcal{L}(\alpha_1) \).
Problem 2

Solution  We first evaluate
\[ \mathcal{L}(\alpha_1) = \mathcal{L}(\emptyset^* \cup a^* \cup b^* \cup a \cup b \cup (a \cup b)^*) \]
\[ = e \cup \{a\}^* \cup \{b\}^* \cup \{a\} \cup \{b\} \cup (\{a\} \cup \{b\})^* = \Sigma^* \]
This is true because of the properties:
\[ (\{a\} \cup \{b\})^* = \{a, b\}^* = \Sigma^* \quad \text{and} \]
\[ \{a\} \subseteq \{a\}^*, \quad \{b\} \subseteq \{b\}^*, \quad \{a\}^* \subseteq \Sigma^*, \quad \{b\}^* \subseteq \Sigma^* \]
and we know that for any sets \( A, B \), if \( A \subseteq B \), then \( A \cup B = B \)
\[ \mathcal{L}(\alpha_1) = \Sigma^* = (\{a\} \cup \{b\})^* = \mathcal{L}((a \cup b)^*) \]
We hence simplify \( \alpha_1 \) as follows
\[ \alpha_1 = \emptyset^* \cup a^* \cup b^* \cup a \cup b \cup (a \cup b)^* = (a \cup b)^* \]
Part 2  Given
\[ \alpha_2 = (a \cup b)^* b (a \cup b)^* \]

Re-write \( \alpha_2 \) as a simpler expression representing the same language

Describe the language \( L = \mathcal{L}(\alpha_2) \)

Solution  \( \alpha_2 \) can not be simplified, but we can use property 
\[ (\{a\} \cup \{b\})^* = \Sigma^* \]  to describe informally the language determined by \( \alpha_2 \) as 
\[ L = \mathcal{L}(\alpha_2) = \Sigma^* b \Sigma^* \]

Remember that informal description \( \Sigma^* b \Sigma^* \) is not a regular expression - but just an useful notation
Problem 3

Let $\Sigma = \{a, b\}$ and a language $L \subseteq \Sigma^*$ be defined as follows:

$$L = \{w \in \Sigma^* : w \text{ contains no more then two } a\text{'s}\}$$

Write a regular expression $\alpha$, such that $L(\alpha) = L$. Use shorthand notation. **Explain** shortly your answer.

**Solution**

$$\alpha = b^* \cup b^* ab^* \cup b^* ab^* ab^*$$

**Explanation**

$\ b^*$ contains 0 of $ a$’s (case $n=0$)

$\ b^*ab^*$ contains 1 occurrence of $ a$ (case $n=1$)

$\ b^*ab^*ab^*$ contains 2 occurrence of $ a$ (case $n=2$)
Problem 4

Let \( \Sigma = \{a, b\} \)
The language \( L \subseteq \Sigma^* \) is defined as follows:
\( L = \{ w \in \Sigma^* : \text{the number of } b \text{'s in } w \text{ is divisible by 4} \} \)

Write a regular expression \( \alpha \), such that \( L(\alpha) = L \)
You can use shorthand notation. Explain shortly your answer

Solution
\( \alpha = a^* (a^* ba^* ba^* ba^* ba^*)^* \)

Observe that the regular expression \( a^* ba^* ba^* ba^* ba^* \)
describes a string \( w \in \Sigma^* \) with exactly four \( b \)'s
Problem 4

The regular expression

\[(a^*ba^*ba^*ba^*ba^*)^*\]

represents multiples of \(w \in \Sigma^*\) with exactly four \(b\)'s and hence words in which a number of \(b\)'s is divisible by 4.

Observe that 0 is divisible by 4, so we need to add the case of 0 number of \(b\)'s, i.e. we need to include words \(e, a, aa, aaa, \ldots\).

We do so by concatenating \(a^*(a^*ba^*ba^*ba^*ba^*)^*\) with \(a^*\) and get

\[L = a^*(a^*ba^*ba^*ba^*ba^*)^*\]
Problem 5

1. Let \( A = \left\{ (\{n, n+1\}, n) \in 2^N \times N : 1 \leq n \leq 3 \right\} \)
List all elements of \( A \)

Solution

1. By simple evaluation we get

\[
A = \left\{ (\{n, n+1\}, n) \in 2^N \times N : n = 1, 2, 3 \right\} \\
   = \{(\{1, 2\}, 1), (\{2, 3\}, 2), (\{3, 4\}, 3)\}
\]
Problem 5

2. Let now \( A = \{ (\{ n \}, n) \in 2^N \times N : \ 1 \leq n \leq n + 1 \} \)

Prove that \( A \) is infinitely countable

Solution

Observe that the set \( A \) can be re-written as follows

\[
A = \{ (\{ n \}, n) \in 2^N \times N : \ 1 \leq n \leq n + 1 \}
\]

\[
= \{ (\{ n \}, n) \in 2^N \times N : \ 1 \leq n \}
\]

because \( n \leq n + 1 \) for all \( n \in N \)

The set \( B = \{ \{ n \} : n \in N \} \) has the same cardinality as \( N \) by the function \( f(n) = \{ n \} \)

\( A = B \times N \) is hence a Cartesian product of two infinitely countable sets, and as we have proved, an infinitely countable set
Problem 6

Let \( L \) be a language defined as follows

\[
L = \{ w \in \{a, b\}^* : P(a, b) \}
\]

for the property \( P(a, b) \) defined as follows

\( P(a, b) : \) between any two \( a \)'s in \( w \) there is an even number of consecutive \( b \)'s

1. **Describe** a regular expression \( r \) such that \( L(r) = L \)

Remark that 0 is an even number, hence \( a^* \in L \) and

\[
r = b^* \cup b^* a^* b^* \cup b^* (a(bb)^*a)^* b^* = b^* a^* b^* \cup b^* (a(bb)^*a)^* b^*
\]
Problem 7

Let $\Sigma$ be any alphabet, $L_1, L_2$ two languages over $\Sigma$ such that $e \in L_1$ and $e \in L_2$

Show that

$$(L_1 \Sigma^* L_2)^* = \Sigma^*$$

Solution

By definition, $L_1 \subseteq \Sigma^*$, $L_2 \subseteq \Sigma^*$ and $\Sigma^* \subseteq \Sigma^*$

Hence

$$(L_1 \Sigma^* L_2) \subseteq \Sigma^*$$
Problem 7

Now we use the following property:

**Property**
For any languages $L_1, L_2$,
if $L_1 \subseteq L_2$, then $L_1^* \subseteq L_2^*$
and obtain that $(L_1 \Sigma^* L_2)^* \subseteq \Sigma^{**} = \Sigma^*$, i.e. we proved that

$$(L_1 \Sigma^* L_2)^* \subseteq \Sigma^*$$

We have to show now that also

$$\Sigma^* \subseteq (L_1 \Sigma^* L_2)^*$$

Let $w \in \Sigma^*$, we have that also $w \in (L_1 \Sigma^* L_2)^*$ because $w = e\varepsilon e$ and $e \in L_1$ and $e \in L_2$. We have hence proved that

$$(L_1 \Sigma^* L_2)^* = \Sigma^*$$
Let $\mathcal{L}$ be a function that associates with any regular expression $\alpha$ the regular language $L = \mathcal{L}(\alpha)$

1. Evaluate $L = \mathcal{L}(\alpha)$ for $\alpha = (a \cup b)^* a$

Solution

$L = \mathcal{L}((a \cup b)^* a) = \mathcal{L}((a \cup b)^*) \mathcal{L}(a) = (\mathcal{L}(a \cup b))^* \{a\} = (\mathcal{L}(a) \cup \mathcal{L}(b))^* \{a\} = (\{a\} \cup \{b\})^* \{a\} = \{a, b\}^* \{a\}$

2. Describe a property that defines the language $L = \mathcal{L}((a \cup b)^* a)$

Solution

$L = \{a, b\}^* \{a\} = \Sigma^* \{a\} = \{w \in \{a, b\}^* : w \text{ ends with } a \}$