cse303
ELEMENTS OF THE THEORY OF COMPUTATION

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LECTURE 4a
REVIEW FOR CHAPTER 1

1. Some Short Questions
2. Some Homework Problems
CHAPTER 1

SHORT QUESTIONS
Short YES/NO Questions

Here are solutions to some short YES/NO Questions for material covered in CHAPTER 1

Solving Quizzes and Tests you have to write a short solutions and circle the answer

You will get 0 pts if you only circle your answer without providing a solution, even if it is correct answer

Here are some questions
Short YES/NO Questions

Q1 \[ \{\emptyset, \{\emptyset\}\} \cap \{\emptyset\} \neq \emptyset \]

yes

We have that

\[ \emptyset \in \{\emptyset\} \quad \text{and} \quad \emptyset \in \{\emptyset, \{\emptyset\}\} \]

This proves that

\[ \emptyset \in \{\emptyset\} \cap \{\emptyset, \{\emptyset\}\} \]

Hence \[ \{\emptyset, \{\emptyset\}\} \cap \{\emptyset\} \neq \emptyset \]
Short YES/NO Questions

Q2  Some relations $R \subseteq A \times B$ are functions that map the set $A$ into the set $B$

yes

Functions are special type of relations so some binary relations are functions (but not all relations are functions)

Q3  $2^\emptyset = \emptyset$

no

$\emptyset \subseteq \emptyset$ so $\emptyset \in 2^\emptyset$
Q4  For any binary relation $R$ on a set $A$, the inverse relation $R^{-1}$ exists.

yes

By definition of the inverse relation is

$$R^{-1} = \{(b, a) : (a, b) \in R\}$$

and such set always exists.
Short YES/NO Questions

Q5  For any function \( f : A \rightarrow B \), the inverse function \( f^{-1} : B \rightarrow A \) exists.
no
Inverse function to a function \( f \) exists if and only if \( f \) is \( 1-1 \) and \( onto \).

Q6  A set \( A = \{ x \in N : x^2 + 5 < 0 \} \) is countable.
yes
\[
A = \{ x \in N : x^2 + 5 < 0 \} = \emptyset
\]
and any finite set is countable.
Short YES/NO Questions

Q7  The set $A = \{ n \in N : n^2 + 5 > 0 \}$ is countable  yes  

The definition says:  
A set $A$ is countable if and only if is finite  or is countably infinite  

The set  

$$A = \{ n \in N : n^2 + 5 > 0 \} = N$$

and $N$ is countably infinite, hence $A$ is countable
Q8 The set \( A = \{ (\{ n \}, n) \in 2^N \times N : 1 \leq n \leq n^2 \} \) is infinitely countable. 

First observe that

\[
A = \{ (\{ n \}, n) \in 2^N \times N : 1 \leq n \leq n^2 \} = C \times B
\]

where the set \( B \) is

\[
B = \{ n \in N : 1 \leq n \leq n^2 \}
\]

and the set \( C \) is

\[
C = \{ \{ n \} \in 2^N : 1 \leq n \leq n^2 \}
\]
Short YES/NO Questions

The condition $1 \leq n \leq n^2$ holds for all $n \in \mathbb{N} - \{0\}$ hence the set

$$B = \{n \in \mathbb{N} : 1 \leq n \leq n^2\}$$

is infinitely countable and so is the set

$$C = \{\{n\} \in 2^{\mathbb{N}} : 1 \leq n \leq n^2\}$$

as the function $f(n) = \{n\}$ is $1 - 1$ and maps $B$ onto $C$.

The set

$$A = C \times B$$

is infinitely countable as it is the cartesian product of two infinitely countable sets.
Q9  Let \( A = \{ n \in N : n^2 + 1 \leq 15 \} \)

It is possible to define 8 alphabets \( \Sigma \subseteq A \)

yes

The set

\[
A = \{ n \in N : n^2 + 1 \leq 15 \} = \{0, 1, 2, 3\}
\]

so the set \( A \) has 4 elements and it has \( 2^4 = 16 \) of all possible subsets and they are all finite, i.e. we can define up to up to 16 alphabets \( \Sigma \subseteq A \)

So have can define for sure 8 < 16 alphabets
Q10 Let $\Sigma = \{ n \in \mathbb{N} : n^2 + 1 = 10 \}$
There are uncountably many finite languages over $\Sigma$

Observe that

$$\Sigma = \{ n \in \mathbb{N} : n^2 + 1 = 10 \} = \{3\}$$

and hence $|\Sigma^*| = \aleph_0$

A finite language over $\Sigma$ is by definition a finite subset of $\Sigma^*$

We have a Theorem:

The set of all finite subsets of any countably infinite set is countably infinite
Short YES/NO Questions

Q11  For any languages \( L_1, L_2, L \) over \( \Sigma \neq \emptyset \) we have that

\[
(L_1 \cup L_2) \cap L = (L_1 \cap L) \cup (L_2 \cap L)
\]

yes  Languages are sets hence all laws of algebra of sets hold for them and this is one of the Distributivity laws

Q12  \( L^* = \{w_1w_2\ldots w_n : w_i \in L, i = 1,2,..n, n \geq 1\} \)

no  This is the definition of \( L^+ \); we must put \( n \geq 0 \) for \( L^* \)
Q13  A regular language is a regular expression
no
A regular language is represented by a regular expression

More precisely, a regular language is represented by the function  $\mathcal{L}$: Regular Expressions $\rightarrow$ Regular Languages
such that the following holds
if $\alpha$ is any regular expression, then $\mathcal{L}(\alpha)$ is the language represented by $\alpha$
Short YES/NO Questions

Q14 Let $\alpha = a(a \cup b)^*$

$\mathcal{L}(\alpha) = \{w \in \{a, b\}^* : w \text{ ends with } a\}$

no

We evaluate

$\mathcal{L}(a(a \cup b)^*) = \{a\} (\{a\} \cup \{b\})^* = \{a\} \Sigma^*$

and hence the property defining $\mathcal{L}(\alpha)$ is

$\mathcal{L}(\alpha) = \{w \in \{a, b\}^* : w \text{ starts with } a\}$
Short YES/NO Questions

Q15  For any language $L$ over an alphabet $\Sigma$,

$$L^+ = L \cup L^*$$

no
Take $L$ be any language such that $e \notin L$
We have that

$$e \notin L^+ \quad \text{but} \quad e \in L \cup L^*$$

This proves that

$$L^+ \neq L \cup L^*$$
CHAPTER 1

Some Homework Problems
Problem 1

Consider the following languages over $\Sigma = \{a, b\}$

$$L_1 = \{w \in \Sigma^* : \exists u \in \Sigma \Sigma (w = uu^R u)\}$$

$$L_2 = \{w \in \Sigma^* : ww = www\}$$

Part 1: Prove that $L_1$ is a finite set

Give example of 3 words $w \in L_1$

Solution

We evaluate first the set $\Sigma \Sigma = \{a, b\} \{a, b\} = \{aa, bb, ab, ba\}$

$\Sigma \Sigma$ is a finite set, hence the set $B = \{uyu : u, y \in \Sigma \Sigma\}$ is also a finite set and by definition $L_1 \subseteq B$

This proves that $L_1$ must be a finite set
Problem 1

We evaluated that  \( \Sigma \Sigma = \{a, b\}\{a, b\} = \{aa, bb, ab, ba\} \)
We defined  \( L_1 = \{w \in \Sigma^* : \exists u \in \Sigma \Sigma (w = uu^R u)\} \)
By evaluation we have that

\[
L_1 = \{aaaaaa, abbaab, baabba, bbbbbb\}
\]

Part 2: Give examples of 2 words over \( \Sigma \) such that \( w \notin L_1 \)
Solution  \( a \notin L_1, \ bba \notin L_1 \)
There are countably infinitely many words that are not in \( L_1 \)
Problem 1

Part 3  Consider now the following language

\[ L_2 = \{ w \in \{a, b\}^* : \; ww = www \} \]

Show that \( L_2 \neq \emptyset \)

Solution  \( e \in L_2 \), as \( ee = eee \)

In fact, \( e \) is the only word with this property, hence

\[ L_2 = \{ e \} \]

Part 4  Show that the set \( (\Sigma^* - L_2) \) is infinite

Solution  \( \Sigma^* \) is countably infinite, \( L_2 \) is finite, so (basic theorem) \( (\Sigma^* - L_2) \) is countably infinite

Any \( w \in \Sigma^* \), such that \( w \neq e \) is in \( (\Sigma^* - L_2) \)
Problem 2

Problem 2
Given expressions (written in a short hand notation)

\[ \alpha_1 = \emptyset^* \cup a^* \cup b^* \cup a \cup b \cup (a \cup b)^* \]

\[ \alpha_2 = (a \cup b)^* b(a \cup b)^* \]

Part 1 Re-write \( \alpha_1 \) as a simpler expression representing the same language

List properties you used in your solution

Describe the language \( L = \mathcal{L}(\alpha_1) \)
Problem 2

Solution  We first evaluate

\[ L(\alpha_1) = L(\emptyset^* \cup a^* \cup b^* \cup a \cup b \cup (a \cup b)^*) \]
\[ = e \cup \{a\}^* \cup \{b\}^* \cup \{a\} \cup \{b\} \cup (\{a\} \cup \{b\})^* = \Sigma^* \]

This is true because of the properties:

\[ (\{a\} \cup \{b\})^* = \{a, b\}^* = \Sigma^* \]

and

\[ \{a\} \subseteq \{a\}^*, \quad \{b\} \subseteq \{b\}^*, \quad \{a\}^* \subseteq \Sigma^*, \quad \{b\}^* \subseteq \Sigma^* \]

and we know that for any sets \( A, B \), if \( A \subseteq B \), then \( A \cup B = B \)

\[ L(\alpha_1) = \Sigma^* = (\{a\} \cup \{b\})^* = L((a \cup b)^*) \]

We hence simplify \( \alpha_1 \) as follows

\[ \alpha_1 = \emptyset^* \cup a^* \cup b^* \cup a \cup b \cup (a \cup b)^* = (a \cup b)^* \]
Problem 3

Part 2 Given

\[ \alpha_2 = (a \cup b)^* b (a \cup b)^* \]

Re-write \( \alpha_2 \) as a simpler expression representing the same language

Describe the language \( L = L(\alpha_2) \)

Solution \( \alpha_2 \) can not be simplified, but we can use property \( (\{a\} \cup \{b\})^* = \Sigma^* \) to describe informally the language determined by \( \alpha_2 \) as

\[ L = L(\alpha_2) = \Sigma^* b \Sigma^* \]

Remember that informal description \( \Sigma^* b \Sigma^* \) is not a regular expression - but just an useful notation
Problem 3

Let $\Sigma = \{a, b\}$ and a language $L \subseteq \Sigma^*$ be defined as follows:

$$L = \{w \in \Sigma^* : w \text{ contains no more then two } a\text{'s}\}$$

Write a regular expression $\alpha$, such that $L(\alpha) = L$. Use shorthand notation. **Explain** shortly your answer.

**Solution**

$$\alpha = b^* \cup b^*ab^* \cup b^*ab^*ab^*$$

**Explanation**

$b^*$ contains 0 of $a$’s (case $n=0$)

$b^*ab^*$ contains 1 occurrence of $a$ (case $n=1$)

$b^*ab^*ab^*$ contains 2 occurrence of $a$ (case $n=2$)
Problem 4

Problem 4
Let $\Sigma = \{a, b\}$
The language $L \subseteq \Sigma^*$ is defined as follows:
$L = \{w \in \Sigma^* : \text{the number of } b \text{'s in } w \text{ is divisible by 4 }\}$
Write a regular expression $\alpha$, such that $L(\alpha) = L$
You can use shorthand notation. Explain shortly your answer

Solution
$\alpha = a^*(a^*ba^*ba^*ba^*ba^*)^*$
Observe that the regular expression $a^*ba^*ba^*ba^*ba^*$ describes a string $w \in \Sigma^*$ with exactly four $b$'s
Problem 4

The regular expression

$$(a^*ba^*ba^*ba^*ba^*)^*$$

represents multiples of $w \in \Sigma^*$ with exactly four $b$ ’s and hence words in which a number of $b$ ’s is divisible by 4. Observe that 0 is divisible by 4, so we need to add the case of 0 number of $b$ ’s, i.e. we need to include words $e, a, aa, aaa, \ldots$.

We do so by concatenating $(a^*ba^*ba^*ba^*ba^*)^*$ with $a^*$ and get

$$L = a^*(a^*ba^*ba^*ba^*ba^*)^*$$
Problem 5

1. Let \( A = \{(\{n, n+1\}, n) \in 2^N \times N : 1 \leq n \leq 3\} \)

List all elements of \( A \)

Solution

1. By simple evaluation we get

\[
A = \{(\{n, n+1\}, n) \in 2^N \times N : n = 1, 2, 3\} \\
= \{(\{1, 2\}, 1), (\{2, 3\}, 2), (\{3, 4\}, 3)\}
\]
Problem 5

2. Let now \( A = \{(\{n\}, n) \in 2^N \times N : 1 \leq n \leq n + 1\} \)

Prove that \( A \) is infinitely countable

Solution

Observe that the set \( A \) can be re-written as follows

\[
A = \{(\{n\}, n) \in 2^N \times N : 1 \leq n \leq n + 1\} \\
= \{(\{n\}, n) \in 2^N \times N : 1 \leq n\}
\]

because \( n \leq n + 1 \) for all \( n \in N \)

The set \( B = \{\{n\} : n \in N\} \) has the same cardinality as \( N \) by the function \( f(n) = \{n\} \)

\( A = B \times N \) is hence a Cartesian product of two infinitely countable sets, and as we have proved, an infinitely countable set
Problem 6

Let $L$ be a language defined as follows

$$L = \{ w \in \{a, b\}^* : P(w) \}$$

for the property $P(w)$ defined as follows

$P(w):$ between any two $a$'s in $w \in \{a, b\}^*$ there is an even number of consecutive $b$'s

1. **Describe** a regular expression $r$ such that $L(r) = L$

Remark that 0 is an even number, hence $a^* \in L$ and

$$r = b^* \cup b^*a^*b^* \cup b^*(a(bb)^*a)^*b^* = b^*a^*b^* \cup b^*(a(bb)^*a)^*b^*$$
Problem 7

Let $\Sigma$ be any alphabet, $L_1, L_2$ two languages over $\Sigma$ such that $e \in L_1$ and $e \in L_2$

Show that

$$(L_1 \Sigma^* L_2)^* = \Sigma^*$$

Solution

By definition, $L_1 \subseteq \Sigma^*$, $L_2 \subseteq \Sigma^*$ and $\Sigma^* \subseteq \Sigma^*$

Hence

$$(L_1 \Sigma^* L_2) \subseteq \Sigma^*$$
Problem 7

Now we use the following property:

**Property**

For any languages $L_1, L_2$, if $L_1 \subseteq L_2$, then $L_1^* \subseteq L_2^*$

and obtain that $(L_1 \Sigma^* L_2)^* \subseteq \Sigma^{**} = \Sigma^*$, i.e. we proved that

$$(L_1 \Sigma^* L_2)^* \subseteq \Sigma^*$$

We have to show now that also

$$\Sigma^* \subseteq (L_1 \Sigma^* L_2)^*$$

Let $w \in \Sigma^*$, we have that also $w \in (L_1 \Sigma^* L_2)^*$ because $w = ewe$ and $e \in L_1$ and $e \in L_2$. We have hence proved that

$$(L_1 \Sigma^* L_2)^* = \Sigma^*$$
Problem 8

1. Evaluate \( L = \mathcal{L}(\alpha) \) for \( \alpha = (a \cup b)^* a \)

Solution

\[
L = \mathcal{L}((a \cup b)^* a) = \mathcal{L}((a \cup b)^*)\mathcal{L}(a) = (\mathcal{L}(a \cup b))^*\{a\} = (\mathcal{L}(a) \cup \mathcal{L}(b))^*\{a\} = (\{a\} \cup \{b\})^*\{a\} = \{a, b\}^*\{a\}
\]

2. Describe a property that defines the language \( L = \mathcal{L}((a \cup b)^* a) \)

Solution

\[
L = \{a, b\}^*\{a\} = \Sigma^*\{a\} = \{w \in \{a, b\}^* : w \text{ ends with } a\}
\]