CSE303 Q4 SOLUTIONS

YES/NO questions

1. The set of terminals is always non- empty
   Justify: Finite set can be empty **n**
2. $L(G) = \{w \in V : S \Rightarrow^*{}_G w\}$
   Justify: $w \in \Sigma^*$ **n**
3. Any regular language is context-free
   Justify: 1. Any Finite Automata is a PDF automata
   2.Regular languages are generated by regular grammars, that are also CF. **y**
4. Language is regular if and only if is generated by a regular grammar (right- linear)
   Justify: proof in class **y**
5. The stack alphabet of a pushdown automaton is always non- empty
   Justify: finite set can be empty **n**
6. $\Delta \subseteq (K \times \Sigma^* \times \Gamma^*) \times (K \times \Gamma^*$ is a transition relation of a pushdown automaton (lecture definition)
   Justify: Δ must be finite **n**
7. $L(M) = \{w \in \Sigma^* : (s, w, e) \models^*{}_M (f, e, e)\}$
   Justify: $f \in F$ **n**
8. Any regular language is accepted by a pushdown automaton
   Justify: Any finite automata is a pushdown automata operating on an empty stock. **y**
9. Context-free languages are closed under intersection
   Justify: Take $L_1 = a^n b^n c^m, L_2 = a^m b^n c^n$, both CF and we get that $L_1 \cap L_2 = a^n b^n c^n$ is not CF **n**
10. The union of a context-free language and regular language is a context-free language
    Justify: regular language is also a context free language and context free languages are closed under union **y**

PROBLEMS

QUESTION 1 Consider a grammar $G = (V, \Sigma, R, S)$, where

$$V = \{a, b, S, A\}, \quad \Sigma = \{a, b\},$$

$$R = \{S \to aB, \ |bA, \ A \to a \ |aS \ |BAA \ B \to b \ |bS \ |ABB\}.$$

1. Show that $ababba \in L(G)$.

Solution The following derivation proves that $ababba \in L(G)$.

$$S \Rightarrow aB \Rightarrow abS \Rightarrow abaB \Rightarrow ababS \Rightarrow ababbA \Rightarrow ababba$$

3. Give a property defining the $L(G)$. Justify shortly your answer.

Case 1 We start with $S \rightarrow aB$. The next steps are applications of rules: $R_1 : B \rightarrow b$, $R_2 : B \rightarrow bS$, or $R_3 : B \rightarrow ABB$. By R_1 we get $ab \in L$, by R_2 we get $aABB$ and A can become a, or aS, or BAA, in all cases we get words with the same number of a's and b's.

Case 2 we start with $S \rightarrow aB$. Th enext step is application of B rules and the "way out". We can see that in this case we will also get words with the same number of a's and b's.

$$L(G) = \{w \in \{a, b\}^* : w\ has\ the\ same\ number\ of\ a's\ and\ b's$$

QUESTION 2 Construct a context-free grammar G such that

$$L(G) = \{wcw^R : w \in \{a, b\}^*\}.$$

Justify your answer.

Solution $\Sigma = \{a, b, c\}, V = \Sigma \cup \{S\}$,

$$R = \{S \rightarrow aSa|bSb|c\}$$

Example of a derivation:

$$S \Rightarrow aSa \Rightarrow abSab \Rightarrow abcba$$

Remark that the set of rules

$$R = \{S \rightarrow aSa|aSb|e\}$$

defines a grammar with

$$L(G) = \{ww^R : w \in \{a, b\}^*\}.$$

QUESTION 3 Construct a context-free grammar G such that

$$L(G) = \{w \in \{a, b\}^* :\ w\ has\ twice\ as\ many\ b's\ as\ a's\}.$$

Justify your answer.

Solution: $G = (V, \Sigma, R, S)$, where $V = \{S, a, b\}, \Sigma = \{a, b\}$.

The set R of rules contains the following rules:

$S \longrightarrow e, \quad S \longrightarrow Sabb|aSbb|abSb|abbbS, \quad S \longrightarrow Sbab|bSab|baSb|babS,$
$S \longrightarrow Sbba|bSba|bbSa|bbaS.$

DEFINITION A context-free grammar $G = (V, \Sigma, R, S)$ is called **regular** iff

$$R \subseteq (V - \Sigma) \times \Sigma^*((V - \Sigma) \cup \{e\})$$

QUESTION 4 Given a **Regular grammar** $G = (V, \Sigma, R, S)$, where

$$V = \{a, b, S, A, B\}, \quad \Sigma = \{a, b\},$$

$$R = \{S \to abA \ |B \ | \ baB \ |e, \quad A \to bS \ |b, \ B \to aS\}.$$

1. Construct a finite automaton M, such that $L(G) = L(M)$. You can draw a diagram.

Solution $M = (K, \Sigma, s, F, \Delta)$ for $K = \{S, A, B, f\}, \Sigma = \{a, b\}, s = S, F = \{f\}$,

$$\Delta = \{(S, ab, A), (S, e, B), (A, ba, B), (S, e, f), (A, b, S), (A, b, f), (B, a, S)\}$$

2. Trace a transitions of M that lead to the acceptance of the string $abba$, and compare with a derivation of the same string in G.

$$(S, abba) \overset{M}{\vdash} (A, ba) \overset{M}{\vdash} (S, a) \overset{M}{\vdash} (B, a) \overset{M}{\vdash} (B, a) \overset{M}{\vdash} (S, e) \overset{M}{\vdash} (f, e)$$

This means that

$$(Sabba) \overset{*,M}{\vdash} (f, e)$$

and $abba \in L(M)$.

The corresponding Derivation in G is:

$$S \Rightarrow abA \Rightarrow abbS \Rightarrow abbB \Rightarrow abbaS \Rightarrow abba$$

QUESTION 5

1. Construct a **pushdown** automaton M such that

$$L(M) = \{w \in \{a, b\}^* : \ w = w^R\}$$

Solution $M = (K, \Sigma, \Gamma, \Delta, s, F)$, where

$$K = \{s, f\}, \quad \Sigma = \{a, b\}, \quad \Gamma = \{a, b\}, \ F = \{f\}, s = s,$$

$$\Delta = \{((s, a, e), (s, a)), \ ((s, b, e), (s, b)), \ ((s, e, e), (f, e)),$$
$$((s, a, e), (f, e)), \ ((s, b, e), (f, e)), \ ((f, a, a), (f, e)), \ ((f, b, b), (f, e))\}$$

2. Trace a transitions of M that lead to the acceptance of the string $ababa$.

Solution

s	$ababa$	e
s	$baba$	a
s	aba	ba
f	ba	ba
f	a	a
f	e	e

QUESTION 6 Show that $L = \{ww : w \in \{a,b\}^*\}$ is not CONTEXT-Free.

Solution We know that the language

$$L_1 = \{a^i b^j a^i b^j : \quad i, j, \geq 0\}$$

is NOT Context Free.

We have a THEOREM :

$$CF \cap REG = CF$$

i.e that intersection of a CF language with a REGULAL language is a CF language.

proof: Assume that L is a CF language. By the Theorem, $L \cap a^* b^* a^* b^*$ must be a CF language, but

$$L \cap a^* b^* a^* b^* = \{a^i b^j a^i b^j : \quad i, j, \geq 0\} = L_1$$

and we have a contradiction.

QUESTION 7 Construct a context-free grammar G such that

$$L(G) = \{w \in \{a,b\}^* : \ w = w^R\}.$$

Justify your answer.

Solution $G = (V, \Sigma, R, S)$, where

$$V = \{a, b, S\}, \quad \Sigma = \{a, b\},$$

$$R = \{S \to aSa \ |bSb \mid a \mid b \mid e\}.$$

Derivation example: $S \Rightarrow aSa \Rightarrow abSba \Rightarrow ababa$
$ababa^R = ((ab)a(ba))^R = (ba)^R a^R (ab)^R = ababa.$

Observation 1 We proved in class that for any $x, y \in \Sigma^*$, $(xy)^R = y^R x^R$. From this we have that

$$(xyz)^R = ((xy)z)^R = z^R(xy)^R = z^R y^R x^R$$

Grammar correctness justification: observe that the rules $S \to aSa \,|bSb \mid e$ generate the language $L_1 = \{ww^R : \; w \in \Sigma^*\}$. With additional rules $S \to a \mid b$ we get hence the language $L = L_1 \cup \{waw^R : \; w \in \Sigma^*\} \cup \{wbw^R : w \in \Sigma^*\}$. Now we are ready to prove that

$$L = L(G) = \{w \in \{a, b\}^* : \; w = w^R\}.$$

Proof Let $w \in L$, i.e. $w = xx^R$ or $w = xax^R$ or $w = xbx^R$. We show that in each case $w = w^R$ as follows.

c1: $w^R = (xx^R)^R = (x^R)^R x^R = xx^R = w$ (used property: $(x^R)^R = x$).

c2: $w^R = (xax^R)^R = (x^R)^R a^R x^R = xax^R = w$ (used Observation 1 and properties: $(x^R)^R = x$ and $a^R = a$).

c3: $w^R = (xbx^R)^R = (x^R)^R b^R x^R = xbx^R = w$ (used Observation 1 and properties: $(x^R)^R = x$ and $b^R = b$).