CSE303 SAMPLE Q1 SOLUTIONS

PART 1: Yes/No Questions Circle the correct answer (each question is worth 1pt.) Write SHORT justification.

1. For any binary relation $R \subseteq A \times A$, we can extend it to $R_1 \subseteq A \times A$, such that R_1 is order on A.

   Justify: Take $R = \{(a,a),(b,b),(a,b),(b,a)\}$. R is an equivalence relation. It is a symmetric relation and can't extend it to R_1, $R \subseteq R_1$ and R_1 antisymmetric.

   n

2. A path (chain) in $R \subseteq A \times A$ is a sequence $a_1...a_n$, for $n \geq 1$ such that $(a_i, a_{i+1}) \in R$ for $i = 1, 2....n-1$. Let $R = \{(a,a),(a,b)\}$. There is a path from a to a in R.

   Justify: Path $a_1 = a, a_2 = a$ (of the length 2). Observe that there is also a path of a length 1 from a to a and path $aaaaaaaaa$ of any length from a to a. **y**

3. Set A is countable iff $N \subseteq A$ (N is the set of natural numbers).

   Justify: $A = \{\emptyset\}$ is countable (finite), but N is not a subset of $\{\emptyset\}$, i.e. $N \not\subseteq \{\emptyset\}$.

   In fact A can be ANY finite set, or any infinite set that does not include N, for example $A = \{\{n\} : n \in N\}$.

   In this case $|A| = |N|$, but N is not a subset of A, i.e. $N \not\subseteq A$. **n**

4. Let $A \neq \emptyset$ such that there are exactly 5 of possible partitions of A. It is possible to define 10 equivalence relations on A.

   Justify: Equivalence relation defines one partition and any partition defines exactly one equivalence relation. So there can be only 5 equivalence relations. **n**

5. 2^N is infinitely countable.

   Justify: $|2^N| = |R| = \mathcal{C}$ and R are uncountable. **n**

6. Let $A = \{\{n\} \in 2^N : n^2 + 1 \leq 15\}$. A is infinite.

   Justify: $\{n \in N : n^2 + 1 \leq 15\} = \{0, 1, 2, 3\}$,
   hence $A = \{\{0\}, \{1\}, \{2\}, \{3\}\}$ is a finite set. **n**

7. Let $\Sigma = \{a\}$. There are countably many languages over Σ.

   Justify: There is any many languages as subsets of $\Sigma^\star$, i.e. uncountably many and exactly as many as real numbers. **n**

8. For any L, $L^+ = L^\star - \{e\}$.

   Justify: Only when $e \notin L$. **n**

9. $L^* = \{w_1 ... w_n : w_i \in L, i = 1, 2, ..n, n \geq 1\}$.
   Justify: $n \geq 0$. **n**
10. For any language L over an alphabet Σ, $L^+ = L \cup L^*$.
    Justify: Take L such that $e \notin L$. We get that $e \in L \cup L^*$ as $e \in L^*$ and $e \notin L^+$. **n**

PART 2: PROBLEMS

QUESTION 1

1. Let $f : A \longrightarrow B$.
Show that the following relation R is an equivalence relation on A.

$$R = \{(a, b) \in A \times A : \quad f(a) = f(b)\}.$$

Solution: R is **reflexive**.
$(a, a) \in R$ for all $a \in A$ because $f(a) = f(a)$ always.
R is **symmetric**, i.e $(a, b) \in R$ means $f(a) = f(b)$, but hence $f(b) = f(a)$ and consequently $(b, a) \in R$.
R is **transitive**, because $f(a) = f(b)$ and $f(b) = f(c)$ implies that $f(a) = f(c)$.

2. Let $f : N \longrightarrow N$, such that

$$f(n) = \begin{cases} 1 & \text{if } n \leq 6 \\ 2 & \text{if } n > 6 \end{cases}$$

Find equivalence classes of R from **1.** for this particular f.

Solution: we evaluate

$$[0] = \{n \in N : f(0) = f(n)\} = \{n \in N : f(n) = 1\} = \{0, 1, 2, 3, 4, 5, 6\}$$

$$[7] = \{n \in N : f(7) = f(n)\} = \{n \in N : f(n) = 2\} = \{n \in N : n > 6\}$$

There are two equivalence classes:

$$A_1 = \{n \in N : n \leq 6\}, \quad A_2 = \{n \in N : n > 6\}.$$

QUESTION 2 Prove by Mathematical Induction:

1. Every order (partial order) on a non-empty finite set has at least one maximal element.

Solution: Let $(A, \preceq)$ be a finite, not empty poset (partially ordered set by $\preceq$). $|A| = n$. Induction over $n \in N - \{0\}$.

Maximal element definition An element $a_o \in A$ ia a maximal element in a poset $(A, \preceq)$ iff the following is true.

$$\neg \exists a \in A(a_0 \neq a \cap a_0 \preceq a).$$

Base case: $n = 1$, $A = \{a\}$. a is maximal (and minimal, and smallest, and largest) in the poset $(\{a\}, \preceq)$.

Inductive step: Assume that a_0 is a maximal element in $(A, \preceq)$, and $|A| = n$. Let B be a set $B = A \cup \{b_0\}$ for $b_0 \notin A$. Of course $|B| = n + 1$. To show that $(B, \preceq)$ has a maximal element we need to consider 3 cases.

(i) $b_0 \preceq a_0$; in this case a_0 is maximal in $(B, \preceq)$.

(ii) $a_0 \preceq b_0$; in this case b_0 is a new maximal in $(B, \preceq)$.

(iii) a_0, b_0 are not compatible; in this case a_0 remains maximal in $(B, \preceq)$.

2. Show that **1** is not true for an infinite set. You can draw a diagram.

Solution: Consider a poset $(Z, \leq)$, where Z is the set on integers and $\leq$ is a natural order on Z. Obviously no maximal element!

QUESTION 3 (4pts) Let Σ be any alphabet, L_1, L_2 two languages over Σ such that $e \in L_1$ and $e \in L_2$. Show that

$$(L_1 \Sigma^\star L_2)^\star = \Sigma^\star$$

Solution: By definition, $L_1 \subseteq \Sigma^\star, L_2 \subseteq \Sigma^\star$ and $\Sigma^\star \subseteq \Sigma^\star$. Hence

$$(L_1 \Sigma^\star L_2) \subseteq \Sigma^\star.$$

Now we use the following property:

Property For any languages $L_1.L_2$,

if $L_1 \subseteq L_2$, then ${L_1}^* \subseteq {L_2}^*$

and obtain that

$$(L_1 \Sigma^\star L_2)^\star \subseteq \Sigma^{\star\star} = \Sigma^\star.$$

We have to show that also

$$\Sigma^\star \subseteq (L_1 \Sigma^\star L_2)^\star.$$

Let $w \in \Sigma^\star$ we have that also $w \in (L_1 \Sigma^\star L_2)^\star$ because $w = ewe$ and $e \in L_1$ and $e \in L_2$.

QUESTION 4 Let $\mathcal{L}$ be a function that associates with any regular expression α the regular language $\mathcal{L}(\alpha)$.

1. Evaluate $\mathcal{L}(((a \cup b)^\star a))$.

Solution: $\mathcal{L}(((a{\cup}b)^\star a)) = \mathcal{L}((a{\cup}b)^\star)\mathcal{L}(a) = (\mathcal{L}(a{\cup}b))^\star\{a\} = (\mathcal{L}(a){\cup}\mathcal{L}(b))^\star\{a\} = (\{a\} \cup \{b\})^\star\{a\} = \{a,b\}^\star\{a\}$.

2. Describe a property that defines the language $\mathcal{L}(((a \cup b)^\star a))$.

Solution $\{a,b\}^\star\{a\} = \{w \in \{a,b\}^\star : \quad w \text{ ends with } a \}$.

QUESTION 5 Let $\Sigma = \{a,b\}$. Let $L_1 \subseteq \Sigma^\star$ be defined as follows:

$$L_1 = \{w \in \Sigma^\star : \text{ number of } b \text{ in } w \text{ is divisible three}\}$$

Write a regular expression α, such that $\mathcal{L}(\alpha) = L_1$. You can use shorthand notation. Explain shortly your answer.

Solution:

$$\alpha = a^\star(a^*ba^\star ba^\star ba^\star)^\star = a^\star(ba^\star ba^\star ba^\star)^\star.$$

Explanation: the part $a^*ba^\star ba^\star ba^\star$ says that there must be 3 occurrences of b in L_1. The part $(a^*ba^\star ba^\star ba^\star)^\star$ says that we the number of b's is $3n$ for $n \geq 1$.

Observe that 0 is divisible by 3, so we need to add the case of 0 number of b's $(n = 0)$, i.e. words $e, a, aa, aaa, , ...$ We do so by concatenating $(a^*ba^\star ba^\star ba^\star)^\star$ with a^*.