CSE303 Q1 SOLUTIONS

PART 1: YES/NO QUESTIONS Circle the correct answer. Write SHORT justification.

1. $2^{\{1,2\}} \cap \{1,2\} \neq \emptyset$
   Justify: $2^{\{1,2\}} = \{\emptyset, \{1\}, \{2\}, \{1,2\}\} \cap \{1,2\} = \emptyset$.

   n

2. Some $R \subseteq A \times B$ are functions that map A into B.
   Justify: Functions are special type of relations.

   y

3. If A is uncountable, then $|A| = |R|$ (R is the set of real numbers).
   Justify: 2^R is uncountable, but $|R| < |2^R|$ by Cantor Theorem.

   n

4. For any function f from A onto A, $f(a) \neq a$.
   Justify: Identity function: $f(x) = x$ for all $x \in A$ maps A onto A.

   n

5. $\{\{a, b\}\} \in 2^{\{a,b,\{a,b\}\}}$
   Justify: $\{\{a, b\}\} \subseteq \{a, b, \{a, b\}\}$.

   y

6. Let $\Sigma = \{n \in N : n \leq 1\}$.
   There are infinitely many finite languages over Σ.

   Justify: $\Sigma = \{0, 1\}$ and $\Sigma^\star$ is countably infinite. The set of all finite subsets of any countably infinite set is countably infinite.

   y

7. $L^+ = \{w_1 ... w_n : w_i \in L, i = 1, 2, ..n, n \geq 1\}$.
   Justify:definition

   y

8. Regular language is a regular expression.
   Justify: Regular Language is represented by the function $\mathcal{L}$

   $$\mathcal{L} : RegExpressions \longrightarrow RegularLanguages$$

   n

9. Every regular language is represented by a regular expression.
   Justify: definition

y

10. For any langauge L over an alphabet Σ, $L^+ = L \cup L^*$.
    Justify: $e \in L^*$ and $e \notin L^+$

n

PART 2: PROBLEMS Solutions

QUESTION 1

1. Let $A = \{(\{n, n+1\}, n) \in 2^N \times N : \ 1 \le n \le 3\}$. List all elements of A.

Solution

$$A = \{(\{n, n{+}1\}, n) \in 2^N \times N : n = 1, 2, 3\} = \{(\{1, 2\}, 1), (\{2, 3\}, 2), (\{3, 4\}, 3)\}$$

2. Let now $A = \{(\{n\}, n) \in 2^N \times N : \quad 1 \le n \le n+1\}$. Prove that A is infinitely countable.

Solution

$$A = \{(\{n\}, n) \in 2^N \times N : \ \ 1 \le n \le n{+}1\} = \{(\{n\}, n) \in 2^N \times N : \ \ 1 \le n\}$$

because $n \le n+1$ for all $n \in N$.
The set $B = \{\{n\} : n \in N\}$ has the same cardinality as N by the function $f(n) = \{n\}$. $A = B \times N$ is a Cartesian product of two infinitely countable sets, and hence is also infinitely countable.

QUESTION 2 Let L_1, L_2 be the following languages over $\Sigma = \{a, b\}$:

$$L_1 = \{w \in \Sigma^\star : \ \ \exists u \in \Sigma\Sigma(w = uu^R u)\},$$

$$L_2 = \{w \in \Sigma^\star : \ \ ww = www\}.$$

1. Give examples 3 words w such that $w \in L_1$. Prove that L_1 is finite.

Solution We evaluate

$$\Sigma\Sigma = \{aa, bb, ab, ba\}$$

$\Sigma\Sigma$ is a finite set, hence the set $B = \{xyx : x, y \in \Sigma\Sigma\}$ is a finite set. $L_1 \subseteq B$, what proves that L_1 must be finite. In fact,

$$L_1 = \{aaaaaa, abbaab, baabba, bbbbbb\}.$$

2. Give examples of 2 words w such that $w \notin L_1$.

Solution $a \notin L_1, bba \notin L_1$. There are countably infinitely many words that are not in L_1.

3. Show that $L_2 \neq \emptyset$.

Solution $e \in L_2$, as $ee = eee$. In fact, e is the only word with this property, hence

$$L_2 = \{e\}.$$

4. Show that the set $(\Sigma^\star - L_2)$ is infinite.

Solution $\Sigma^\star$ is countably infinite, L_2 is finite, so (basic theorem) $(\Sigma^\star - L_2)$ is countably infinite. Any $w \in \Sigma^\star$, such that $w \neq e$ is in $(\Sigma^\star - L_2)$.

QUESTION 3 Given expressions (written in a short hand notation)

$$\alpha_1 = \emptyset^\star \cup a^\star \cup b^\star \cup a \cup b \cup (a \cup b)^\star, \qquad \alpha_2 = (a \cup b)^\star b (a \cup b)^\star$$

1. Re-write α_1 as a simpler expression representing the same language. List properties you used in your solution. Describe the language.

Solution We use the following properties:

(i) $\{a\} \subseteq \{a\}^\star$, $\{b\} \subseteq \{b\}^\star$,

(ii) $\{a\} \cup \{b\} = \{a, b\}$, $\{a\} \subseteq \{a, b\}$, $\{b\} \subseteq \{a, b\}$.

From the above and property 1. from the Question 2, $\{a\}^\star \subseteq \{a, b\}^\star$, $\{b\}^\star \subseteq \{a, b\}^\star$, and of course, $\emptyset \subset A$ for any set A and, if $A \subseteq B$, then $A \cup B = B$.

Translating the properties above into regular expressions we get that

$$\alpha_1 = (a \cup b)^\star.$$

The language described by α_1 is

$$\mathcal{L}(\alpha_1) = \Sigma^\star.$$

2. Re-write α_2 as a simpler expression representing the same language. Describe the language.

Solution α_2 can not be simplified. We use only property that $(\{a\} \cup \{b\})^\star = \Sigma^\star$ to describe the language determined by α_2

$$\mathcal{L}(\alpha_2) = \Sigma^\star b \Sigma^\star.$$

QUESTION 4 Let $\Sigma = \{a, b\}$. Let $L_1 \subseteq \Sigma^\star$ be defined as follows:

$$L_1 = \{w \in \Sigma^\star : \text{ the number of } b\text{'s in } w \text{ is divisible by } 4\}.$$

Write a regular expression α, such that $\mathcal{L}(\alpha) = L_1$. You can use shorthand notation. Explain shortly your answer.

Solution

$$\alpha = a^\star(a^\star ba^\star ba^\star ba^\star ba^\star)^\star = a^\star(ba^\star ba^\star ba^\star ba^\star)^\star$$

Observe that the regular expression $(a^\star ba^\star ba^\star ba^\star ba^\star)^\star$ describes a string $w \in \Sigma^\star$ with exactly 4 b's.

The regular expression $(a^\star ba^\star ba^\star ba^\star ba^\star)^\star$ represents multiples of w, and hence words in which a number of b's is divisible by 4.

Observe that 0 is divisible by 4, so we need to add the case of 0 number of b's ($n = 0$), i.e. words $e, a, aa, aaa, , ...$ We do so by concatenating $(a^\star ba^\star ba^\star ba^\star ba^\star)^\star$ with a^*.