1 YES/NO questions (10pts)

Circle the correct answer (each question is worth 1pts) Write SHORT justification. No justification - no points

1. All infinite sets have different cardinality.
   Justify: The sets \( N \) (natural numbers) and \( Z \) (integers) have the same cardinality: \( |N| = |Z| = \aleph_0 \). This is not the only example. n

2. Regular expression defines a regular language.
   Justify: By definition: "A language \( L \) is regular iff there is a regular expression \( \alpha \) such that \( L = L(\alpha) \). y

3. \( L^+ = \{w_1...w_n : w_i \in L, i = 1,2..., n \geq 2\} \).
   Justify: the correct condition is \( n \geq 1 \). n

4. \( L^+ = L^* - \{e\} \).
   Justify: It holds only when \( e \notin L \). When \( e \in L \) we get that \( e \in L^+ \) and \( e \notin L^* - \{e\} \). n

5. Let \( \alpha = (\emptyset^* \cap b^*) \cup \emptyset^* \). The language defined by \( \alpha \) is empty.
   Justify: \( L = L(\alpha) = (\{e\} \cap \{b\}^*) \cup \{e\} = \{e\} \cup \{e\} = \{e\} \neq \emptyset \) n

6. A configuration of any finite automaton \( M = (K, \Sigma, \Delta, s, F) \) is any element of \( K \times \Sigma^* \times K \).
   Justify: it is element of \( K \times \Sigma^* \) n

7. Let \( M \) be a finite state automaton, \( L(M) = \{\omega \in \Sigma^* : (s,\omega) \xrightarrow{s,M} (q,e)\} \).
   Justify: only when \( q \in F \) n

8. For any \( M, L(M) \neq \emptyset \) if and only if the set \( F \) of its final states is non-empty.
   Justify: Let \( M \) be such that \( \Sigma = \emptyset, F \neq \emptyset, s \notin F \), we get \( L(M) = \emptyset \). n

9. The set \( F \) of final states of any non-deterministic finite automaton is always non-empty.
   Justify: The definition says that \( F \) is a finite set, i.e. can be empty, hence for some \( M \), \( L(M) = \emptyset \). n

10. DFA and NDFA recognize the same class of languages.
    Justify: Equivalency theorem proved in class y

2 Two definitions of a non-deterministic automaton

BOOK DEFINITION: \( M = (K, \Sigma, \Delta, s, F) \) is non-deterministic when
\[ \Delta \subseteq K \times (\Sigma \cup \{e\}) \times K \]

OBSERVE that \( \Delta \) is always finite because \( K, \Sigma \) are finite sets.

LECTURE DEFINITION: \( M = (K, \Sigma, \Delta, s, F) \) is non-deterministic when \( \Delta \) is finite and
\[ \Delta \subseteq K \times \Sigma^* \times K \]

OBSERVE that we have to say in this case that \( \Delta \) is finite because \( \Sigma^* \) is infinite.

SOLVING PROBLEMS you can use any of these definitions.
3 Very Short Questions (15pts)

For all Questions below do the following.

1. Draw the State Diagram.
2. Determine whether it defines a finite state automaton.  3. Determine whether it is/it is not an automaton, it is/it is not a deterministic / non-deterministic automaton
4. Describe the language by writing a regular expression that defines it.

Q1 (5pts) \[ M = (K, \Sigma, s, \Delta, F) \] for \[ K = \{q_0\} = F, \ s = q_0, \Sigma = \emptyset, \Delta = \emptyset. \]

Solution \[ M \] is deterministic and \[ L(M) = \{e\} \neq \emptyset \]

Q2 (5pts) \[ M = (K, \Sigma, s, \Delta, F) \] for \[ \Sigma = \{a, b\}, K = \{q_0, q_1\}, s = q_0, F = \{q_0\}, \Delta = \{(q_0, a, q_1), (q_1, b, q_0)\}. \]

Solution \[ M \] is non deterministic; \[ \Delta \] is not a function on \[ K \times \Sigma. \]

\[ L(M) = (ab)^* \]

Q3 (5pts) \[ M = (K, \Sigma, s, \Delta, F) \] for \[ \Sigma = \{a, b\}, K = \{q_0, q_1, q_2\}, F = \{q_1\}, \Delta = \{(q_0, a, q_1), (q_0, b, q_1), (q_1, a, q_1), (q_1, e, q_2), (q_2, ab, q_2)\}. \]

Solution It is NOT an automaton. It has no initial state.

4 PROBLEMS (50pts + 15extra)

PROBLEM 1 (20pts)

Given an automaton \[ M = (K, \Sigma, \delta, s, F) \] such that \[ \Sigma = \{a, b\}, \ K = \{q_0, q_1, q_2, q_3\}, \ s = q_0, \ F = \{q_0\} \] and \[ q_3 \] is a trap state.

We define \[ \delta \] on non-trap states as follows.

\[ \delta(q_0, a) = q_1, \ \delta(q_0, b) = q_0, \ \delta(q_1, a) = q_2, \ \delta(q_1, b) = q_1, \ \delta(q_2, a) = q_0 \]

1. (10pts) DRAW a complete state diagram of \[ M. \]

Solution

Draw the diagram for

\[ \delta(q_0, a) = q_1, \ \delta(q_0, b) = q_0, \ \delta(q_1, a) = q_2, \ \delta(q_1, b) = q_1, \ \delta(q_2, a) = q_0 \]

with \[ \delta \] on the trap state \[ q_3 \] defined below.

2. (2pts) DEFINE formally \[ \delta \] for the trap state

Solution We define \[ \delta \] on the trap state \[ q_3 \] as follows.

\[ \delta(q_2, b) = q_3, \ \delta(q_3, a) = q_3, \ \delta(q_3, b) = q_3 \]

3. (2pts) LIST 4 elements of \[ L(M) \] and 4 elements not in \[ L(M) \]

Solution list your own elements - here are mine.

\[ e, \ bbb, \ abbaa, \ abbaaabbaa \in L(M), \ \text{and} \ a, \ aa, \ aba, \ bab \notin L(M) \]

4. (3pts) WRITE a regular expression defining the language \[ L(M) \] of \[ M. \]
Solution: Directly from the diagram we can "read" that the language of $M$ is
\[ L(M) = b^* \cup (b^*ab^*aab^*)^* \]

5. (3pts) JUSTIFY your answer.

Solution: The initial state $q_0$ is also a final state so we get $b^* \in L(M)$ as a loop on $q_0$ and the fact that in this case also $eL(M)$

Obviously $b^*ab^*aab^* \in L(M)$ and then all repetitions of it are in $L(M)$ as a loop on $q_0$.

PROBLEM 2 (10pts)

Let $M = (K, \Sigma, s, \Delta, F)$ for $K = \{q_0\}$, $s = q_0$, $\Sigma = \{a, b\}$, $F = \{q_0\}$ and
\[ \Delta = \{(q_0, aba, q_0), (q_0, ab, q_0)\} \]

1. (3pts) Draw the State Diagram of $M$.

Solution: just do it!

2. (2pts) List 4 elements of $L(M)$.

Solution: For example $e$, $ab$, $abab$, $ababaaba$, ...

3. (2pts) WRITE a regular expression defining the language accepted by $M$.

Solution: The language is $L = (ab \cup aba)^*$.

4. (3pts) JUSTIFY why your expression is correct.

Solution: The initial state is a finite state hence $e \in L$. Of course for the same reason $aba \in L$ or $ab \in L$ and so are all possible combinations of both, and $L = (ab \cup aba)^*$.

PROBLEM 3 (20pts)

Let $M$ be defined as follows
\[ M = (K, \Sigma, s, \Delta, F) \]
for $K = \{q_0, q_1, q_2, q_3\}$, $s = q_0$,
\[ \Sigma = \{a, b, c\}, \quad F = \{q_0, q_2, q_3\} \]
and
\[ \Delta = \{(q_0, abc, q_0), (q_0, a, q_1), (q_0, c, q_3), (q_0, bc, q_1), (q_1, b, q_2), (q_2, a, q_2), (q_2, b, q_3), (q_3, a, q_3)\} \]

1. (2pts) Draw the state diagram of $M$.

Solution: just do it!

2. (3pts) Find the regular expression describing the $L(M)$.

Solution: You can "read" directly from the diagram of $M$ that
\[ L(M) = (abc)^* \cup (abc)^*ab^* \cup (abc)^*a(abc)^*ba^* \cup (abc)^*a(abc)^*ba^* \cup (abc)^*a^* \]

If you need to provide explanation - here is mine:

$\alpha_1 = (abc)^*$ - loop on $q_0$,
$\alpha_2 = (abc)^*a(abc)^*ba^*$ - path from $q_0$ to $q_2$,
$\alpha_3 = (abc)^*a(abc)^*ba^*ba^*$ - path from $q_0$ to $q_3$ via $q_2$. 

3
\[ \alpha_4 = (abc)^*a^* - \text{path from } q_0 \text{ directly to } q_3 \]

Observe that \( e \in L \) as \( q_0 \in F \) and also \((q_0, e, q_3) \in \Delta \) and \( q_3 \in F \).

3. (10 pts) DRAW the diagram an automaton \( M' \) such that \( M' \equiv M \) and \( M' \) is defined by the BOOK definition.

**Solution** just draw components listed below

4. (5 pts) List all components of \( M' \).

**Solution**

We apply the "stretching" technique to \( M \) and the new \( M' \) COMPONENTS are as follows.

\[ M' = (K \cup \{p_1, p_2, p_3\}, \Sigma, s = q_0, \, \Delta', \, F' = F) \]

for \( K = \{q_0, q_1, q_2, q_3\}, \, s = q_0 \)
\( \Sigma = \{a, b\}, \, F' = \{q_0, q_2, q_3\} \) and
\( \Delta' = \{(q_0, a, q_1), (q_0, e, q_3), (q_1, b, q_2), (q_2, a, q_2), (q_2, b, q_3), (q_3, a, q_3)\}\)
\( \{(q_0, a, p_1), (p_1, b, p_2), (p_2, c, q_0), (q_1, b, p_3), (p_3, b, q_1)\}\)

**EXTRA CREDIT** (15pts)

For \( M \) defined as follows
\[ M = (K, \Sigma, s, \Delta, F) \]

for \( K = \{q_0, q_1, q_2, q_3\}, \, s = q_0, \, \Sigma = \{a, b\}, \, F = \{q_2, q_3\} \) and
\( \Delta = \{(q_0, a, q_1), (q_0, e, q_1), (q_0, b, q_2), (q_1, b, q_3), (q_1, e, q_3), (q_2, b, q_2), (q_3, e, q_3), (q_3, a, q_3)\} \)

1. (2 pts) Draw the State Diagram of \( M \).

**Solution**

2. (10 pts) Write 4 steps of the general method of transformation a NDFA \( M \), into an equivalent \( M' \), which is a DFA.

**Reminder:** \( E(q) = \{p \in K : (q, e) \xrightarrow{\ast M} (p, e)\} \) and
\[ \delta(Q, \sigma) = \bigcup \{E(p) : \exists q \in Q, \, (q, \sigma, p) \in \Delta\}. \]

**Step 1** (2 pts) Evaluate \( E(q) \) for all \( q \in K \).

**Solution**
\[ E(q_0) = \{q_0, q_1, q_3\}, \, E(q_1) = \{q_1, q_3\}, \, E(q_2) = \{q_2, q_3\}, \, E(q_3) = \{q_4\} \]

**Step 2** (2 pts) Evaluate \( \delta(E(q_0), a) \) and \( \delta(E(q_0), b) \).

**Solution**
\[ \delta(E(q_0), a) = \delta(\{q_0, q_1, q_3\}, a) = E(q_1) \cup E(q_3) = \{q_1, q_3\} \in F, \]
\[ \delta(E(q_0), b) = \delta(\{q_0, q_1, q_3\}, b) = E(q_2) \cup E(q_3) \cup \emptyset = \{q_2, q_3\} \in F, \]

**Step 3** (3 pts) Evaluate \( \delta\) on all states that result from step 2.
Solution

\[ \delta(\{q_1, q_3\}, a) = \emptyset \cup E(q_3) = \{q_3\} \in F, \]

\[ \delta(\{q_1, q_3\}, b) = E(q_3) \cup \emptyset = \{q_3\} \in F, \]

\[ \delta(\{q_2, q_3\}, a) = \emptyset \cup E(q_3) = \{q_3\} \in F, \]

\[ \delta(\{q_2, q_3\}, b) = E\{q_2\} \cup \emptyset = \{q_2, q_3\} \in F \]

**Step 4** (3pts) Evaluate \( \delta \) on all states that result from step 3.

Solution

\[ \delta(\{q_3\}, a) = E(q_3) = \{q_3\} \in F, \quad \delta(\{q_3\}, b) = \emptyset, \]

\[ \delta(\emptyset, a) = \emptyset, \quad \delta(\emptyset, b) = \emptyset \]

**End** of the construction.

3. (3pts) Draw the DIAGRAM of \( M' \) after the steps you have **finished** evaluating.

Solution just draw it.