Test consists of 2 parts: YES/NO Questions (15pts), Problems (50pts).

The extra 5pts are included in the TOTAL 65 pts sum for the whole test.

PART 1 (15pts) YES/NO QUESTIONS - (1pts) each

Circle the correct answer. Write SHORT justification. Answers without justification will not receive credit.

1. \( L(M_1) = L(M_2) \) iff \( M_1 \) and \( M_2 \) are finite automata.
   \textbf{Justify:} take as \( M_1 \) any automata such that \( L(M_1) \neq \emptyset \) and \( M_2 \) such that \( L(M_2) = \emptyset \)

2. A language is regular if and only if \( L = L(M) \) and \( M \) is a finite automaton
   \textbf{Justify:} Main Theorem

3. Every subset of a regular language is a language.
   \textbf{Justify:} subset of a set is a set and languages are sets

4. Any finite language is CF.
   \textbf{Justify:} any finite language is regular and \( RL \subseteq CFL \)

5. Intersection of any two regular languages is CF language.
   \textbf{Justify:} Regular languages are closed under intersection and \( RL \subseteq CFL \)

6. \( L \) is regular if and only if there is a CF grammar \( G \), such that \( L = L(G) \).
   \textbf{Justify:} \( G \) with \( R = \{ S \rightarrow aSb | e \} \) is CF but the \( L(G) = \{ a^n b^n : n \geq 0 \} \) is NOT Regular

7. Let \( \Sigma = \{ a \} \), then for any \( w \in \Sigma^*, w^R = w \)
   \textbf{Justify:} \( a^R = a \) and \( w^R = w \) for \( w \in \{ a \}^* \)

8. Let \( G = (\{ S,(,) \},\{(,), R,S \}) \) for \( R = \{ S \rightarrow SS | (S) \} \). \( L(G) \) is regular.
   \textbf{Justify:} \( L(G) = \emptyset \) and hence regular

9. If \( L \) is regular, then there is a CF grammar \( G \), such that \( L = L(G) \).
   \textbf{Justify:} \( RL \subseteq CF \)

10. A CF language is a regular language.
    \textbf{Justify:} \( L = \{ a^n b^n : n \geq 0 \} \) is CF and not regular

11. The set of terminals in a context free grammar \( G \) is a subset of alphabet of \( G \)
    \textbf{Justify:} \( \Sigma \subseteq V \)

12. \( L(G) = \{ w \in V : S \Rightarrow_G w \} \)
    \textbf{Justify:} \( w \in \Sigma^* \)
13. \( L = \{a^n a^n : n \geq 0\} \) is not regular.
   **Justify:** \( L = (aa)^* \) and hence is regular.

14. \( L = \{a^n b^n c^n : n \geq 0\} \) is Context-Free.
   **Justify:** \( L \) is not CF, as proved by Pumping Lemma for CF languages.

15. \( L \subseteq \Sigma^* \) is context-free if and only if \( L = L(G) \).
   **Justify:** \( G \) must be a context-free grammar.

16. \( L(G) = \{ w \in \Sigma : S \Rightarrow^* G w \} \).
   **Justify:** \( w \in \Sigma^* \).

**PART 2: PROBLEMS (50 pts)**

**Problem 1 (5 pts)**

Let \( M = (K, \Sigma, s, \Delta, F) \) for \( K = \{q_0, q_1, q_2\}, s = q_0, \Sigma = \{a, b, c\}, F = \{q_1, q_2\} \) and
\( \Delta = \{(q_0, abc, q_0), (q_0, ab, q_1), (q_0, b, q_2)\} \).

1. (1pt) **Draw** the diagram of \( M \)

   **Solution** - just draw the diagram

2. (4pts) **Draw** the diagram of an automaton \( M' \) such that \( M' \equiv M \) and \( M' \) is defined by the BOOK definition.

   **Solution**

   Here are components for the correct Diagram of \( M' \)

   Please draw the correct diagram yourself

   We apply the "stretching" technique to \( M \) and the new \( M' \) is as follows.

   \[
   M' = (K \cup \{p_1, p_2, p_3\}, \Sigma, s = q_0, \Delta', F' = F)
   \]

   \[
   \Delta' = \{(q_0, a, p_1), (p_1, b, p_2), (p_2, c, q_0), (q_0, a, p_3), (p_3, b, q_1), (q_0, b, q_2)\}
   \]

**Problem 2 (15 pts)**

Evaluate regular expression \( r \), such that \( L(M) = r \) using the Generalized Automata

\( GM = (K_G, \Sigma_G, \Delta_G, s_G, F_G) \) for \( M \) given by
\( M = (\{q_1, q_2\}, \{a, b\}, s = q_1, \Delta = \{(q_1, a, q_1), (q_1, b, q_2), (q_2, a, q_2), (q_2, b, q_1)\}, F = \{q_2\}) \)

1. (3pt) **Draw** a diagram of \( GM \) and list all components. **Remember** using to use the proper names for states for \( GM \)

   **Remark** By definition \( \Sigma_G = \Sigma \cup \mathcal{R}_0 \), where \( \mathcal{R}_0 \) is FINITE subset of regular expressions over \( \Sigma \) constructed by states elimination of the initial \( M \). When writing components of the needed sequence of Generalized Automata equivalent to \( M \), write just \( \Sigma_G = \Sigma \), without specifying the set of \( \mathcal{R}_0 \). The specific elements of \( \mathcal{R}_0 \) appear in in \( \Delta_G \) at each stage.
Solution

MG Diagram (1pt) - just draw it

MG Components (2pt)

The components of GM are as follows

\[ GM = (\{q_1, q_2, q_3, q_4\}, \{a, b\}, s = q_3, F = \{q_4\}) \]
\[ \Delta = \{(q_1, a, q_1), (q_1, a, q_2), (q_2, b, q_2), (q_2, a, q_1), (q_3, c, q_1), (q_2, c, q_4)\} \]

2. (5pt) Draw a diagram of \( GM^1 \simeq GM \simeq M \) obtained by elimination of \( q_1 \). List all components.

Solution

MG1 Diagram (2pt) - draw it

MG1 Components (3pt) are:

\[ GM^1 = (\{q_2, q_3, q_4\}, \{a, b\}, s = q_3, F = \{q_4\}) \]
\[ \Delta = \{(q_3, a^*b, q_2), (q_2, a, q_2), (q_2, ba^*b, q_2), (q_2, e, q_4)\} \]

3. (5pt) Draw a diagram of \( GM^2 \simeq GM^1 \simeq GM \simeq M \) obtained by elimination of \( q_2 \). List all components.

Solution

MG2 Diagram (2pt) - draw it

MG2 Components (3pt)

The components of GM2 are:

\[ GM^2 = (\{q_3, q_4\}, \{a, b\}, s = q_3, F = \{q_4\}) \]
\[ \Delta = \{(q_3, a^*b, qa^*b, q_4)\} \]

4. (2pt) Write the regular expression \( r \), such that \( L(M) = r \)

Solution

\[ L(M) = a^*b(ba^*b \cup a)^* \]

Problem 3 (10 pts)

Use the constructions defined in the proof of the theorem:

A language is regular if and only if it is accepted by a finite automata
to construct a finite automata \( M \) such that

\[ L(M) = (ab \cup c)^* \]

Draw PATTERN diagrams.

Use the constructions described in the proof of the Closure Theorem.

S1. (5pt) Draw diagrams of automata \( M_a, M_b, M_c \) and \( M_aM_b \cup M_c \).

Solution Follow Lecture 7 diagrams for CLOSURE Theorem Proof - and Examples.

S2. (5pt) Draw diagram of \( M = (M_aM_b \cup M_c)^* \).
Problem 4 (5 pts)

1. (2pts) Write derivation of words \textit{ababa} in \( G = (V, \Sigma, R, S) \), where
\( V = \{a, b, S\}, \quad \Sigma = \{a, b\}, \quad R = \{S \rightarrow aSa \mid bSb \mid a \mid b \mid e\}. \)

Solution format: \( S \Rightarrow .... S \Rightarrow ababa \)

Solution
Derivation of \textit{ababa} \( S \Rightarrow aSa \Rightarrow abSba \Rightarrow ababa \)

Derivation of \textit{aababaa} \( S \Rightarrow aSa \Rightarrow aaSaa \Rightarrow aabSbaa \Rightarrow aababaa \)

2. (2pts) Write a proof that the word \textit{ababa} \( \in L(G) \) fulfills a property \( w = w^R \)

Solution item || We know that \( (xy)^R = y^Rx^R \) and so \( (xyz)^R = z^R(xy)^R = z^Ry^R(xy)^R \)

We evaluate \( (ababa)^R = ((ab)a(ba))^R = (ba)^R a^R(ab)^R = ababa \)

3. (1pts) Describe the \( L(G) \)

Solution
\( L(G) = \{w \in \{a, b\}^* : w = w^R\} \)

Problem 5 (15 pts)

Given a Regular grammar \( G = (V, \Sigma, R, S) \), where
\( V = \{a, b, S, A\}, \quad \Sigma = \{a, b\}, \)
\( R = \{S \rightarrow aS \mid A \mid e, \quad A \rightarrow abA \mid a \mid b\}. \)

1. (5pts)

Use the construction in the proof of L-G Theorem:

\textit{Language L is regular if and only if there exists a regular grammar G such that L = L(G) to construct a finite automaton M, such that L(G) = L(M).}

Draw a diagram of M

Solution

We construct a non-deterministic finite automata
\( M = (K, \Sigma, \Delta, s, F) \)

as follows:
\( K = (V - \Sigma) \cup \{f\}, \quad \Sigma = \Sigma, s = S, \quad F = \{f\}, \)
\( \Delta = \{(S, a, S), (S, e, A), (S, e, f), (A, ab, A), (A, a, f), (A, b, f)\} \)
The Diagram is:

![Diagram]

2. (5pts) Trace a transition of $M$ that leads to the acceptance of the string $aaaababa$

Solution format: $(s,aaaababa) \vdash_M$ ....

Solution

The accepting computation is:

$$
(S,aaaababa) \vdash_M (S,aaababa) \vdash_M (S,ababa) \vdash_M (S,ababa) \vdash_M (A,ababa) \\
\vdash_M (A,aba) \vdash_M (A,a) \vdash_M (f,e)
$$

3. (5pts) Trace a derivation of the same string $aaaababa$ in the gramma $G$.

Solution format: $S \Rightarrow$ .... $S \Rightarrow aaaababa$

Solution

The $G$ derivation is:

$$
S \Rightarrow aS \Rightarrow aaS \Rightarrow aaaS \Rightarrow aaaA \Rightarrow aaaabA \Rightarrow aaaababA \Rightarrow aaaababa
$$