Test consists of three parts: YES/NO Questions (15pts), Very Short Questions (15pts), and Problems (30pts). The extra (10pts) are included in the TOTAL 70 pts sum for the whole test.

PART 1: YES/NO QUESTIONS (15pts)

Circle your answer and write a short justification (1pt). No justification - no points

1. \{2, \{0\}, 0\} ∩ 2^0 = ∅
   Justify: \(2^0 = \{0\}\) and \(2, \{0\}, 0\} ∩ \{0\} ≠ ∅ as \(0 ∈ 2, \{0\}, 0\)

2. A set \(A = \{(n, m) ∈ N × N : n^2 < 5, m^2 < 6\}\) is countable
   Justify: \(A = \{(n, m) ∈ N × N : n = 0, 1, 2 \ \text{and} \ m = 0, 1, 2\}\) is a FINITE set, hence countable

3. Set \(A\) is countable if and only if \(|A| = |N|\), where \(N\) is the set of Natural numbers
   Justify: This is definition of infinitely countable set.
   Definition of a countable set is: a set \(A\) is countable iff is finite or infinitely countable.

4. The set \(\{0, \{0\}\}\) is an alphabet
   Justify: Alphabet is any FINITE set and this is a finite set of two elements

5. \(L^* = \{w_1...w_n : w_i ∈ L, i = 1, 2, ..n, n ≥ 0\}\)
   Justify: Definition of Kleene Star operation on languages

6. \(L^* = LL^*\)
   Justify: The problem is only with cases \(e ∈ L\) or \(e ∉ L\).
   When \(e ∈ L\), then \(e ∈ L^*\), and always \(e ∈ L\), and \(e ∈ LL^*\).
   When \(e ∉ L\), then \(e ∉ L^*\), and always \(e ∈ L\), but then \(e ∉ LL^*\) as \(xe = x\) for \(x ∉ e\), hence \(L^* ≠ LL^*\)

7. A regular language is a regular expression
   Justify: A language \(L ⊆ \Sigma^*\) is regular if and only if \(L\) is represented by a regular expression, i.e. when there is \(a ∈ R\) such that \(L = \mathcal{L}(a)\)

8. \((\emptyset^* ∩ a) ∪ b^*\) ∩ \(\emptyset^*\) describes a language with only one element
   Justify: we have that \(\emptyset^* = \{e\}\), \((\emptyset^* ∩ a) = \{e\} ∩ \{a\} = ∅\), and so \((\emptyset^* ∩ a) ∪ b^*\) ∩ \(\emptyset^* = (∅ ∪ \{b^*\}) ∩ \{e\} = \{b^*\} ∩ \{e\} = \{e\}\)

9. \(L^* = \{w_1...w_n : w_i ∈ L, i = 1, 2, ..n, n ≥ 2\}\)
   Justify: the correct condition is \(n ≥ 1\)

10. \(L^* = L^* - \{e\}\)
    Justify: It holds only when \(e ∉ L\). When \(e ∈ L\) we get that \(e ∈ L^*\) and \(e ∉ L^* - \{e\}\)

11. Let \(M\) be a finite state automaton, \(L(M) = \{ω ∈ Σ^* : (s, ω) \overset{*}{→} (q, e)\}\)
    Justify: only when \(q ∈ F\)

12. For any \(M, L(M) ≠ ∅\) if and only if the set \(F\) of its final states is non-empty
    Justify: Let \(M\) be such that \(Σ = ∅, F = ∅, s ∉ F\), we get \(L(M) = ∅\)

13. The set \(F\) of final states of any non-deterministic finite automaton is always non-empty
    Justify: The definition says that \(F\) is a finite set, i.e. can be empty, hence for some \(M, L(M) = ∅\)
14. Alphabet $\Sigma$ of any deterministic finite automaton is always non-empty

**Justify:** An alphabet $\Sigma$ is any FINITE set, hence it can be empty

15. Given an automaton $M = (K, \Sigma, \delta, s, F)$, a binary relation $r_M \subseteq (K \times \Sigma^*) \times (K \times \Sigma^*)$ is a **one step computation** only when the following condition holds:

$(q, aw) r_M (q', w)$ if and only if $\delta(q, a) = q'$

**Justify:** The correct statement is: $(q, aw) r_M (q', w)$ if and only if $\delta(q, a) = q'$

**PART 2:** Very Short Questions (15pts) - 5pts each

For all Questions below do the following.

1. Draw the State Diagram.
2. Determine whether it is/it is not an automaton, it is/it is not a deterministic / non-deterministic automaton
3. Describe the language by writing a regular expression that defines it.

**Q1:** $M$ has components: $K = \{q\}$, $s = q$, $\Sigma = \emptyset$, $\delta = \emptyset$, $F = \{q\}$

**M Diagram** - draw it

**M** is DFA

**Language** of $M$ is $L = \{e\}$ because $s \in F$

**Q2:** $M$ has components: $\Sigma = \{a, b\}$, $K = \{q_0, q_1\}$, $s = q_0$, $F = \{q_0, q_1\}$ and $\delta(q_0, a) = q_0$, $\delta(q_0, b) = q_0$, $\delta(q_1, a) = q_0$, $\delta(q_1, b) = q_1$

**M Diagram** is: just draw it

**M** is DFA

**Language** of $M$ is $L = (a \cup b)^* = \Sigma^*$, $q_1$ is a trap state

**Q3:** $M$ has components $K = \{q_0, q_1, q_2\}$, $\Sigma = \{a, b\}$, $F = \{q_1, q_2\}$.

$\Delta = \{(q_0, a, q_1), (q_0, b, q_2), (q_1, a, q_1), (q_1, b, q_2), (q_2, ab, q_2)\}$

**Diagram** $M$ just draw it

$M$ is NOT an automaton It does not have the INITIAL state.

$L(M)$ does not exist - as languages are defined only for for automata

**PART 3:** PROBLEMS (40pts)

**PROBLEM 1** (10pts)

Given an alphabet $\Sigma = \{a, b, c\}$ and a regular expression $\alpha = ((c^* a) \cup b)$

1. (2pts) Write $\alpha$ in the shorthand notation

   $c^* a \cup b$

2. (4pts) List elements of the infinite set $L = \alpha$ (short notation)

   $[e, c, cc, ccc, ...,][a] \cup [b] = \{a, ca, cca, ccca, ...,\} \cup \{b\} = \{b, a, ca, cca, ccca, ....\}$

3. (4pts) Write a property $P(w)$ describing $L = \alpha$, i.e. write the language $L$ as $L = \{w \in \Sigma^* : P(w)\}$

   $L = \{w \in \Sigma^* : w = a \text{ or } w = b \text{ or } w \text{ is any nonempty finite sequence of } c^* \text{'s followed by a } \}$
PROBLEM 2 (15pts)

Given an automaton

\[ M = (K, \Sigma, \delta, s, F) \]

such that \( \Sigma = \{a, b\} \), \( K = \{q_0, q_1, q_2, q_3\} \), \( s = q_0 \), \( F = \{q_0\} \) and \( q_3 \) is a trap state.

We define \( \delta \) on non-trap states as follows.

\[ \delta(q_0, a) = q_1, \quad \delta(q_0, b) = q_0, \quad \delta(q_1, a) = q_2, \quad \delta(q_1, b) = q_1, \quad \delta(q_2, a) = q_0 \]

1. (8pts) DEFINE \( \delta \) on the trap state \( q_3 \)

Solution We define \( \delta \) on the trap state \( q_3 \) as follows.

\[ \delta(q_2, b) = q_3, \quad \delta(q_3, a) = q_3, \quad \delta(q_3, b) = q_3 \]

2. (3pts) DRAW the complete diagram of \( M \).

Solution Draw the diagram for

\[ \delta(q_0, a) = q_1, \quad \delta(q_0, b) = q_0, \quad \delta(q_1, a) = q_2, \quad \delta(q_1, b) = q_1, \quad \delta(q_2, a) = q_0 \delta(q_2, b) = q_3, \quad \delta(q_3, a) = q_3, \quad \delta(q_3, b) = q_3 \]

3. (4pts) LIST 4 elements of \( L(M) \) and 4 elements not in \( L(M) \)

Solution list your own elements - here are mine.

\[ e, \ bbb, \ abbaa, \ abbaabbaa \in L(M), \ \text{and} \ a, \ aa, \ aba, \ baba \notin L(M) \]

PROBLEM 3 (15pts)

Let \( M = (K, \Sigma, s, \Delta, F) \) for \( K = \{q_0\} \), \( s = q_0 \), \( \Sigma = \{a, b\} \), \( F = \{q_0\} \) and

\[ \Delta = \{(q_0, aba, q_0), (q_0, ab, q_0)\} \]

1. (4pts) Draw the State Diagram of \( M \).

Solution just do it

2. (4pts) List 4 elements of \( L(M) \).

Solution For example \( e, \ ab, \ abab, \ ababa, \ ababaaba, \ ... \)

3. (7pts) WRITE a regular expression defining the language accepted by \( M \).

Solution The language is \( L = (ab \cup aba)^* \)

Justification why the regular expression is correct: the initial state is also a finite state, hence \( e \in L \).

Of course for the same reason \( aba \in L \) or \( ab \in L \) and so are all possible combinations of both, hence \( L = (ab \cup aba)^* \)