CSE303 Q1 SOLUTIONS Spring 2023
25 points + 5 extra pts

NAME SOLUTIONS

PART 1: YES/NO QUESTIONS (15pts)
Circle your answer and write a short justification (1pt). No justification - no points

1. \( \{2, \{0\}, 0\} \cap 2^0 = \emptyset \)

Justify: \( \emptyset = \{0\} \) and \( \{2, \{0\}, 0\} \cap \{0\} \neq \emptyset \) as \( \emptyset \in \{2, \{0\}, 0\} \)

2. For any non-empty sets \( A, B \), any function that maps \( A \) into \( B \) is a binary relation \( R \subseteq A \times B \)

Justify: Functions are, by definition, special binary relations

3. A set \( A = \{ (n, m) \in N \times N : n^2 < 5, m^2 < 6 \} \) is countable

Justify: \( A = \{ (n, m) \in N \times N : n = 0, 1, 2 \ and \ m = 0, 1, 2 \} \) is a FINITE set, hence countable

4. A set \( A = \{(n, m) \in 2^N \times N : n^2 + 1 < 5, m > 6 \} \) is countable

Justify: \( A = \{ (n, m) \in 2^N \times N : n = 0, 1, m > 6 \} = B \times C \)
for \( B = \{0\}, \{1\} \) and \( C = \{m \in N : m > 6\} \). Both sets are countable as B is finite and C infinitely countable and so is their Cartesian product

5. Set \( A \) is countable if and only if \( |A| = |N| \), where \( N \) is the set of Natural numbers.

Justify: This is definition of infinitely countable set.
Definition of a countable set is: a set \( A \) is countable iff it is finite or infinitely countable.

6. If the set \( A \) is uncountable, then \( |A| = |R| \) (\( R \) is the set of real numbers)

Justify: Definition of a un countable set is: \( A \) is countable iff \( A \) is not countable.
For example \( A = 2^R \) is uncountable, but \( |R| < 2^R \) by Cantor Theorem.

7. The set \( \{0, \{0\}\} \) is an alphabet

Justify: \( A \) has two elements so it is a finite set.
8. Let $\Sigma = \emptyset$. There is only one language $L = \emptyset$ over $\Sigma$

**Justify:** $\Sigma^* = \emptyset^* = \{e\}$. We have that and $\emptyset \subseteq \{e\}$ and $\{e\} \subseteq \{e\}$.
Hence there are two languages $L_1 = \emptyset$ and $L_1 = \{e\}$ over $\Sigma$

9. For any languages $L_1, L_2$, $L$ over $\Sigma \neq \emptyset$

$$(L_1 \cup L_2) \cap L = (L_1 \cap L) \cup (L_2 \cap L)$$

**Justify:** Languages are sets, so the LAWS of distributivity hold

10. Let $\Sigma = \{a, b\}$. There are more than 20 words of length 4 over $\Sigma$.

**Justify:** By Counting Functions Theorem there are exactly $2^4 = 16$ words of length 4 over $\Sigma$ and $16 < 20$.

11. For any languages $L_1, L_2$, $L$ over $\Sigma \neq \emptyset$

$$\Sigma^* - (L_1 \cup L_2) = (\Sigma^* - L_1) \cap (\Sigma^* - L_2)$$

**Justify:** Languages are sets so de Morgan Laws hold for them.

12. $L^* = \{w_1...w_n : w_i \in L, \, i = 1, 2, ..n, \, n \geq 0\}$

**Justify:** Definition of Kleene Star operation on languages

13. $L^* = LL^*$.

**Justify:** the problem is only with cases $e \in L$ or $e \notin L$.
When $e \in L$, then $e \in L^*$, and always $e \in L^*$, and $e \in LL^*$.
When $e \notin L$, then $e \notin L^*$, but then $e \notin LL^*$ as $xe = x$ for $x \neq e$, hence $L^* = LL^*$

14. A regular language is a regular expression.

**Justify:** A language $L \subseteq \Sigma^*$ is regular if and only if $L$ is represented by a regular expression, i.e. when there is $\alpha \in \mathcal{R}$ such that $L = L(\alpha)$

15. $(0^* \cap a) \cup b^*) \cap 0^*$ describes a language with only one element.

**Justify:** we have that $0^* = \{e\}$, $(0^* \cap a) = \{e\} \cap \{a\} = \emptyset$, and so
$(0^* \cap a) \cup b^*) \cap 0^* = (0 \cup b^*) \cap \{e\} = \{b\} \cap \{e\} = \{e\}$
PART 2: PROBLEMS (15pts)

QUESTION 1 (8pts)

Given an alphabet $\Sigma = \{a, b, c\}$ and a regular expression $\alpha = (c^*a \cup b)$

1. (1pt) Write $\alpha$ in the shorthand notation $c^*a \cup b$

2. (3pts) Given the representation function $L : \mathcal{R} \rightarrow 2^\Sigma$.

Use the definition of the function $L$ to evaluate, step by step $L = L(\alpha)$.

$L = L(\alpha) = L(c^*a \cup b) = L(c^*a) \cup L(b) = L(c^*)L(a) \cup \{b\} = \{c\}^* \cup \{b\}$

3. (2pts) List elements of the infinite set $L = \alpha$ (short notation)

$$\{e, c, cc, ccc, .... \} \cup \{b\} = \{a, ca, cca, ccca, .... \} \cup \{b\} = \{b, a, ca, cca, ccca, .... \}$$

4. (2pts) Write a property $P(w)$ describing $L = \alpha$, i.e. write the language $L$ as $L = \{w \in \Sigma^* : P(w)\}$

$L = \{w \in \Sigma^* : w = a \text{ or } w = b \text{ or } w \text{ is any nonempty finite sequence of } c\text{'s followed by } a\}$

PROBLEM 2 (7pts)

Let $\Sigma = \{0, 1\}$. Let $L \subseteq \Sigma^*$ be defined as follows.

$L = \{w \in \{0, 1\}^* : \text{w has three occurrences of 1 the first and the second of which are not consecutive }\}$

1. (4pts) Write a regular expression $\alpha$, such that $L = \alpha$ (short notation)

$L = 0^*10^*010^*10^*$

2. (3pts) Explain carefully our answer.

$L$ obviously has three occurrences of 1.

The first and the second occurrences of 1 are not consecutive because in the expression 10*01 the 0 separates them.