1 YES/NO questions

1. For any binary relation $R \subseteq A \times A$, $R^*$ exists.
   
   Justify: definition

2. For any binary relation $R \subseteq A \times A$, $R^{-1}$ exists.
   
   Justify: The set $R^{-1} = \{(b,a) : (a,b) \in R\}$ always exists.

3. For any function $f$ from $A \neq \emptyset$ onto $A$, $f$ has property $f(a) \neq a$ for certain $a \in A$.
   
   Justify: $f(x) = x$ is always "onto".

4. All infinite sets have the same cardinality.
   
   Justify: $|N| < |2^N|$ by Cantor Theorem and $N, 2^N$ are infinite

5. Set $A$ is uncountable iff $R \subseteq A$ ($R$ is the set of real numbers).
   
   Justify: $R, 2^R$ are both uncountable and $R$ is not a subset of $2^R$ ($R \not\subseteq 2^R$) but $R \in 2^R$. 

6. Let $A \neq \emptyset$ such that there are exactly 25 partitions of $A$. It is possible to define 20 equivalence relations on $A$.
   
   Justify: one can define up to 25 (as many as partitions) of equivalence classes

7. There is a relation that is equivalence and order at the same time.
   
   Justify: equality relation

8. Let $A = \{n \in N : n^2 + 1 \leq 15\}$. It is possible to define 8 alphabets $\Sigma \subseteq A$.
   
   Justify: $A$ has 4 elements, so we have $2^4 > 8$ subsets

9. There is exactly as many languages over alphabet $\Sigma = \{a\}$ as real numbers.
   
   Justify: $|\Sigma^*| = \aleph_0, |2^{\Sigma^*}| = |R| = C$. 

10. Let $\Sigma = \{a, b\}$. There are more than 20 words of length 4 over $\Sigma$.
   
   Justify: There are exactly $2^4 = 16$ words of length 4 over $\Sigma$ and 16 < 20. 

11. $L^* = \{w_1...w_n : w_i \in L, i = 1, 2,..n, n \geq 1\}$.
    
    Justify: $n \geq 0$.
    
    $L^+ = LL^*$.
    
    Justify: the problem is only with cases $e \in L$ or $e \not\in L$. When $e \in L$, then $e \in L^+$, and always $e \in L^*$, hence $e \in LL^*$.
    
    When $e \not\in L$, then $e \not\in L^+$, and always $e \in L^*$, hence $e \in LL^*$ and $L^+ \neq LL^*$

12. $L^+ = L^* - \{e\}$.
    
    Justify: only when $e \not\in L$. When $e \in L$ we get that $e \in L^+$ and $e \not\in L^* - \{e\}$. 

13. If \( L = \{ w \in \{0, 1\}^* : w \) has an unequal number of 0’s and 1’s \}, then \( L^* = \{0, 1\}^* \).

\textbf{Justify:} \( 1 \in L, 0 \in L \) so \( \{0, 1\} \subseteq L \subseteq \Sigma^* \), hence \( \{0, 1\}^* \subseteq L^* \subseteq (\Sigma^*)^* = \Sigma^* = \{0, 1\}^* \) and \( L^* = \{0, 1\}^* \).

14. For any languages \( L_1, L_2, (L_1 \cup L_2) \cap L_1 = L_1 \).

\textbf{Justify:} languages are sets and \( (A \cup B) \cap A = A \).

15. For any languages \( L_1, L_2 \),

\[ L_1^* = L_2^* \text{ iff } L_1 = L_2 \]

\textbf{Justify:} Consider \( L_1 = \{a, e\}, L_2 = \{a\} \). Obviously, \( L_1 \neq L_2 \) and \( L_1^* = L_2^* \).

16. For any languages \( L_1, L_2, (L_1 \cup L_2)^* = L_1^* \).

\textbf{Justify:} languages are sets so it is true only when \( L_1 \subseteq L_2 \).

17. \( ((\emptyset^* \cap a) \cup b^*) \cap \emptyset^* \) describes a language with only one element.

\textbf{Justify:} \( \emptyset \cup b^* = b^*, b^* \cap \{e\} = \{e\} \)

18. \( ((\emptyset^* \cap a) \cup b^*) \cap a^* \) is a finite regular language.

\textbf{Justify:} \( b^* \cap a^* = \{e\} = \emptyset^* \)

19. \( \{a\} \cup \{e\} \cap \{ab\}^* \) is a finite regular language.

\textbf{Justify:} \( \{a\} \cup \{e\} \cap \{ab\}^* = \{a, e\} \cap \{ab\}^* = \{e\} = \emptyset^* \)

20. Any regular language has a finite description.

\textbf{Justify:} by definition \( L = L(r) \) and \( r \) is a finite string.

21. Any finite language is regular.

\textbf{Justify:} \( L = \{w_1\} \cup \ldots \cup \{w_1\} \) and \( \{w_3\} \) has a finite description \( w_i \)

22. Every deterministic automata is also non-deterministic.

\textbf{Justify:} \( K \times \Sigma \subseteq K \times \Sigma \cup \{e\} \subseteq \Sigma^* \) and any function is a relation

The set of all configurations of any non-deterministic state automata is always non-empty.

\textbf{Justify:} \( K \neq \emptyset \), because \( s \in K \). If \( \Sigma = \emptyset \) the set of all configuration of non-deterministic automata (book definition) is a subset of \( K \times \emptyset \cup \{e\} \neq \emptyset \) as it always contains \( (s, e) \). For the lecture definition, the set of all configuration is a subset of \( K \times \Sigma^* \) and always \( e \in \Sigma^* \) hence always \( (s, e) \in K \times \Sigma^* \).

23. Let \( M \) be a finite state automaton, \( L(M) = \{ w \in \Sigma^* : (q, w) \xrightarrow{M} (s, e) \} \).

\textbf{Justify:} \( L(M) = \{ w \in \Sigma^* : \exists q \in F((s, w) \xrightarrow{M} (q, e)) \} \)

24. For any automata \( M \), \( L(M) \neq \emptyset \).

\textbf{Justify:} if \( F = \emptyset \), \( L(M) = \emptyset \)

25. \( L(M_1) = L(M_2) \) iff \( M_1, M_2 \) are deterministic.

\textbf{Justify:} Let \( M_1 \) be an automata over \( \{a, b\} \) with with \( \Delta = \{ (q_0, ab, q_0) \} \), \( F = \{q_0\} \), \( s = q_0 \) and let \( M_2 \) be an automata over \( \{a, b\} \) with with \( \Delta = \{ (q_0, ab, q_0), (q_0, e, q_1) \} \), \( F = \{q_1\} \), \( s = q_0 \).

\( L(M_1) = L(M_2) = (ab)^* \) and both are non-deterministic
26. DFA and NDFA compute the same class of languages.

Justify: basic theorem

27. Let $M_1$ be a deterministic, $M_2$ be a nondeterministic FA, $L_1 = L(M_1)$ and $L_2 = L(M_2)$ then there is a deterministic automaton $M$ such that $L(M) = (L^* \cup (L_1 - L_2)^*)L_1$

Justify: the class of finite automata is closed under $\ast, \cup, -, \cap$

TWO DEFINITIONS OF NON DETERMINISTIC AUTOMATA

BOOK DEFINITION: $M = (K, \Sigma, \Delta, s, F)$ is non-deterministic when

$$\Delta \subseteq K \times (\Sigma \cup \{e\}) \times K$$

OBSERVE that $\Delta$ is always finite because $K, \Sigma$ are finite sets.

LECTURE DEFINITION: $M = (K, \Sigma, \Delta, s, F)$ is non-deterministic when $\Delta$ is finite and

$$\Delta \subseteq K \times \Sigma^* \times K$$

OBSERVE that we have to say in this case that $\Delta$ is finite because $\Sigma^*$ is infinite.

SOLVING PROBLEMS you can use any of these definitions.

2 Problems

In Problems 1-5 solutions you have to DRAW DIAGRAMS; do not need to LIST the Components

Problem 1

Let $L$ be a language defined as follows

$$L = \{w \in \{a,b\}^*: \text{every } a \text{ is either immediately preceded or followed by } b\}.$$

1. Describe a regular expression $r$ such that $L(r) = L$ (Meaning of $r$ is $L$).

Solution $L = (b \cup ab \cup ba \cup bab)^*$

2. Construct a finite state automata $M$, such that $L(M) = L$.

Solution

Components of $M$ are:

$$K = \{s\}, \{a,b\}, \ s, \ F = \{s\},$$

$$\Delta = \{(s,b,s), (s,ab,s), (s,ba,s), (s,bab,s)\}$$

Some elements of $L(M)$ are: $b, bb, baab, abab, abbbba, bbbabbbabbab$

Problem 2

1. Let $M = (K, \Sigma, \delta, s, F)$ be a deterministic finite automaton. Under exactly what conditions $e \in L(M)$?

Solution

$$e \in L(M) \text{ iff } s \in F.$$
2. Let $M = (K, \Sigma, \Delta, s, F)$ be a non-deterministic finite automaton. Under exactly what conditions $e \in L(M)$?

**Solution** Now we have two possibilities: $s \in F$ (computation of length 0) or there is a computation of length $> 0$ from $(s, e)$ to $(q, e)$ for $q \in F$ when $s \notin F$.

**Problem 3** Let

$$M = (K, \Sigma, s, \Delta, F)$$

for $K = \{q_0, q_1, q_2, q_3\}$, $s = q_0$

$\Sigma = \{a, b\}$, $F = \{q_1, q_2, q_3\}$ and

$$\Delta = \{(q_0, a, q_1), (q_0, b, q_3), (q_1, a, q_2), (q_1, b, q_1), (q_3, a, q_3), (q_3, b, q_2)\}$$

1. List some elements of $L(M)$.

**Solution** $a, b, aa, bb, aba, abba$

2. Write a regular expression for the language accepted by $M$. Simplify the solution.

**Solution**

$$L(M) = ab^* \cup ab^* a \cup ba^* b = ab^*(e \cup a) \cup ba^*(e \cup b).$$

3. Define a deterministic $M'$ such that $M \approx M'$, i.e. $L(M) = L(M')$.

**Solution** We complete $M$ do a deterministic $M'$ by adding a TRAP state $q_4$ and put

$$\Delta' = \delta = \Delta \cup \{(q_2, a, q_4), (q_2, b, q_4), (q_4, a, q_4), (q_4, b, q_4)\}$$

**Justify** why $M \approx M'$.

**Solution** $q_4$ is a trap state, it does not influence $L(M)$.

**Problem 4** Let $M$ be defined as follows

$$M = (K, \Sigma, s, \Delta, F)$$

for $K = \{q_0, q_1, q_2, q_3\}$, $s = q_0$

$\Sigma = \{a, b, c\}$, $F = \{q_0, q_2, q_3\}$ and

$$\Delta = \{(q_0, abc, q_0), (q_0, a, q_3), (q_0, e, q_3), (q_1, bc, q_1), (q_1, b, q_2), (q_2, a, q_2), (q_2, b, q_3), (q_3, a, q_3)\}.$$ 

Find the regular expression describing the $L(M)$. Explain your steps. Does $e \in L(M)$?

**Solution**

$$L = \alpha_1 \cup \alpha_2 \cup \alpha_3 \cup \alpha_4$$

where

$$\alpha_1 = (abc)^* - \text{loop on } q_0,$$

$$\alpha_2 = (abc)^*a(bc)^*ba^* - \text{path from } q_0 \text{ to } q_2,$$

$$\alpha_3 = (abc)^*a(bc)^*ba^*ba^* - \text{path from } q_0 \text{ to } q_3 \text{ via } q_2,$$

$$\alpha_3 = (abc)^*a^* - \text{path from } q_0 \text{ directly to } q_3$$

This is not the only solution.

Observe that $e \in L$ as $q_0 \in F$ and also $(q_0, e, q_3) \in \Delta$ and $q_3 \in F$.

This is not the only solution.
Write down (you can draw the diagram) an automata $M'$ such that $M' \equiv M$ and $M'$ is defined by the BOOK definition.

Solution

Solution We apply the "stretching" technique to $M$ and the new $M'$ is as follows.

$$M' = (K \cup \{p_1, p_2, p_3\} \Sigma, s = q_0, \Delta', F' = F)$$

for $K = \{q_0, q_1, q_2\}$, $s = q_0$

$\Delta = \{a, b\}$, $F = \{q_0, q_2, q_3\}$ and

$\Delta' = \{(q_0, a, q_1), (q_0, e, q_3), (q_1, b, q_2), (q_2, a, q_2), (q_2, b, q_3), (q_3, a, q_3)\}$

$\cup\{(q_0, a, p_1), (p_1, b, p_2), (p_2, c, q_0), (q_1, b, p_3), (p_3, b, q_1)\}$.

Problem 5 For $M$ defined as follows

$$M = (K, \Sigma, s, \Delta, F)$$

for $K = \{q_0, q_1, q_2, q_3\}$, $s = q_0$

$\Sigma = \{a, b\}$, $F = \{q_2\}$ and

$\Delta = \{(q_0, a, q_3), (q_0, e, q_3), (q_0, b, q_1), (q_0, e, q_1), (q_1, a, q_2), (q_2, b, q_3), (q_2, e, q_3)\}$

Write 2 steps of the general method of transformation the NDFA $M$ defined above into an equivalent DFA $M'$.

Step 1: Evaluate $\delta(E(q_0), a)$ and $\delta(E(q_0), b)$.

Step 2: Evaluate $\delta$ on all states that result from step 1.

**Reminder:** $E(q) = \{p \in K : (q, e) \xrightarrow{\delta} (p, e)\}$ and

$$\delta(q, \sigma) = \bigcup\{E(p) : \exists q \in Q, (q, \sigma, p) \in \Delta\}.$$

Solution Step 1: First we need to evaluate $E(q)$, for all $q \in K$.

$E(q_0) = \{q_0, q_1, q_3\} = S$, $E(q_1) = \{q_1\}$, $E(q_2) = \{q_2, q_3\} \in F$, $E(q_3) = \{q_3\}$

$$\delta(E(q_0), a) = \delta(\{q_0, q_1, q_3\}, a) = E(q_3) \cup E(q_2) \cup \emptyset = \{q_2, q_3\} \in F$$

$$\delta(E(q_0), b) = \delta(\{q_0, q_1, q_3\}, b) = E(q_1) \cup \emptyset \cup \emptyset = \{q_1\}$$

Solution Step 2:

$$\delta(\{q_2, q_3\}, a) = \emptyset \cup \emptyset = \emptyset$$

$$\delta(\{q_2, q_3\}, b) = E(q_3) \cup \emptyset = \{q_3\}$$

$$\delta(\{q_1\}, a) = E(q_2) = \{q_2, q_3\} \in F$$

$$\delta(\{q_1\}, b) = \emptyset$$