CSE303
PRACTICE MIDTERM SOLUTIONS

1 YES/NO questions

1. For any binary relation $R \subseteq A \times A$, R^* exists.
   Justify: definition **y**

2. For any binary relation $R \subseteq A \times A$, R^{-1} exists.
   Justify: The set $R^{-1} = \{(b,a) : \ (a,b) \in R\}$ always exists. **y**

3. For any function f from $A \neq \emptyset$ onto A, f has property $f(a) \neq a$ for certain $a \in A$.
   Justify: $f(x) = x$ is always "onto". **n**

4. All infinite sets have the same cardinality.
   Justify: $|N| < |2^N|$ by Cantor Theorem and $N, 2^N$ are infinite **n**

5. Set A is uncountable iff $R \subseteq A$ (R is the set of *real* numbers).
   Justify:$R, 2^R$ are both uncountable and R is not a subset of 2^R ($R \not\subseteq 2^R$) but $R \in 2^R$. **n**

6. Let $A \neq \emptyset$ such that there are exactly 25 partitions of A. It is possible to define 20 equivalence relations on A.
   Justify: one can define up to 25 (as many as partitions) of equivalence classes **y**

7. There is a relation that is equivalence and *order* at the same time.
   Justify: equality relation **y**

8. Let $A = \{n \in N : n^2 + 1 \leq 15\}$. It is possible to define 8 *alphabets* $\Sigma \subseteq A$.
   Justify:A has 4 elements, so we have $2^4 > 8$ subsets **y**

9. There is exactly as many languages over alphabet $\Sigma = \{a\}$ as real numbers.
   Justify: $|\Sigma^*| = \aleph_0, |2^{\Sigma^*}| = |R| = C$. **y**

10. Let $\Sigma = \{a, b\}$. There are more than 20 words of length 4 over Σ.
    Justify: There are exactly $2^4 = 16$ words of length 4 over Σ and $16 < 20$. **n**

11. $L^* = \{w_1 ... w_n : w_i \in L, i = 1, 2, ..n, n \geq 1\}$.
    Justify: $n \geq 0$. **n**

    $L^+ = LL^*$.

    Justify: the problem is only with cases $e \in L$ or $e \notin L$. When $e \in L$, then $e \in L^+$, and always $e \in L^*$, hence $e \in LL^*$.
    When $e \notin L$, then $e \notin L^+$, and always $e \in L^*$, hence $e \in LL^*$ and $L^+ \neq LL^*$ **n**

12. $L^+ = L^* - \{e\}$.
    Justify: only when $e \notin L$. When $e \in L$ we get that $e \in L^+$ and $e \notin L^* - \{e\}$. **n**

13. If $L = \{w \in \{0,1\}^* :$ w has an unequal number of 0's and 1's $\}$, then $L^* = \{0,1\}^*$.

    Justify: $1 \in L, 0 \in L$ so $\{0,1\} \subseteq L \subseteq \Sigma^*$, hence $\{0,1\}^* \subseteq L^* \subseteq (\Sigma^*)^* = \Sigma^* = \{0,1\}^*$ and $L^* = \{0,1\}^*$. **y**

14. For any languages L_1, L_2, $(L_1 \cup L_2) \cap L_1 = L_1$.

    Justify: languages are sets and $(A \cup B) \cap A = A$. **y**

15. For any languages L_1, L_2,

$$L_1^* = L_2^* \;\; iff \;\; L_1 = L_2$$

    Justify: Consider $L_1 = \{a, e\}, L_2 = \{a\}$. Obviously, $L_1 \neq L_2$ and $L_1^* = L_2^*$. **n**

16. For any languages L_1, L_2, $(L_1 \cup L_2)^* = L_1^*$.

    Justify: languages are sets so it is true only when $L_1 \subseteq L_2$. **n**

17. $((\emptyset^* \cap a) \cup b^*) \cap \emptyset^*$ describes a language with only one element.

    Justify: $\emptyset \cup b^* = b^*, b^* \cap \{e\} = \{e\}$ **y**

18. $((\emptyset^* \cap a) \cup b^*) \cap a*$ is a finite regular language.

    Justify: $b^* \cap a^* = \{e\} = \emptyset^*$ **y**

19. $(\{a\} \cup \{e\}) \cap \{ab\}^*$ is a finite regular language.

    Justify: $(\{a\} \cup \{e\}) \cap \{ab\}^* = \{a, e\} \cap \{ab\}^* = \{e\} = \emptyset^*$ **y**

20. Any regular language has a finite description.

    Justify: by definition $L = \mathcal{L}(r)$ and r is a finite string. **y**

21. Any finite language is regular.

    Justify: $L = \{w_1\} \cup ... \cup \{w_1\}$ and $\{w_i\}$ has a finite description w_i **y**

22. Every deterministic automata is also non-deterministic.
    Justify: $K \times \Sigma \subseteq K \times \Sigma \cup \{e\} \subseteq \Sigma^*$ and any function is a relation **y**

    The set of all configurations of any non-deterministic state automata is always non-empty.

    Justify: $K \neq \emptyset$, because $s \in K$. If $\Sigma = \emptyset$ the set of all configuration of non-deterministic automata (book definition) is a subset of $K \times \emptyset \cup \{e\} \neq \emptyset$ as it always contains (s, e). For the lecture definition, the set of all configuration is a subset of $K \times \Sigma^*$ and always $e \in \Sigma^*$ hence always $(s, e) \in K \times \Sigma^*$ **y**

23. Let M be a finite state automaton, $L(M) = \{w \in \Sigma^* : (q, w) \overset{*,M}{\longmapsto} (s, e)\}$.

    Justify: $L(M) = \{w \in \Sigma^* : \exists q \in F((s, w) \overset{*,M}{\longmapsto} (q, e))\}$ **n**

24. For any automata M, $L(M) \neq \emptyset$.

    Justify: if $F = \emptyset$, $L(M) = \emptyset$ **n**

25. $L(M_1) = L(M_2)$ iff M_1, M_2 are deterministic.

    Justify: Let M_1 be an automata over $\{a, b\}$ with with $\Delta = \{(q_0, ab, q_0)\}, F = \{q_0\}, s = q_0$ and let M_2 be an automata over $\{a, b\}$ with with $\Delta = \{(q_0, ab, q_0), (q_0, e, q_1)\}, F = \{q_1\}, s = q_0$. $L(M_1) = L(M_2) = (ab)^*$ and both are non-deterministic **n**

26. DFA and NDFA compute the same class of languages.
Justify: basic theorem **y**

27. Let M_1 be a deterministic, M_2 be a nondeterministic FA, $L_1 = L(M_1)$ and $L_2 = L(M_2)$ then there is a deterministic automaton M such that $L(M) = (L^* \cup (L_1 - L_2)^*)L_1$
Justify: the class of finite automata is closed under $*, \cup, -, \cap$ **y**

TWO DEFINITIONS OF NON DETERMINISTIC AUTOMATA

BOOK DEFINITION: $M = (K, \Sigma, \Delta, s, F)$ is non-deterministic when

$$\Delta \subseteq K \times (\Sigma \cup \{e\}) \times K$$

OBSERVE that Δ is always finite because K, Σ are finite sets.

LECTURE DEFINITION: $M = (K, \Sigma, \Delta, s, F)$ is non-deterministic when Δ is finite and

$$\Delta \subseteq K \times \Sigma^* \times K$$

OBSERVE that we have to say in this case that Δ is finite because Σ^* is infinite.

SOLVING PROBLEMS you can use any of these definitions.

2 Problems In Problems 1-5 solutions you have to DRAW DIAGRAMS; do not need to LIST the Components

Problem 1 Let L be a language defines as follows

$$L = \{w \in \{a, b\}^* : every\ a\ is\ either\ immediately\ proceeded\ or\ followed\ by\ b\}.$$

1. Describe a regular expression r such that $\mathcal{L}(r) = L$ (Meaning of r is L).

Solution $L = (b \cup ab \cup ba \cup bab)^*$

2. Construct a *finite state automata* M, such that $L(M) = L$.

Solution

Components of M are:

$$K = \{s\}, \{a, b\}, \quad s, \quad F = \{s\},$$

$$\Delta = \{(s, b, s), (s, ab, s), (s, ba, s), (s, bab, s).\}$$

Some elements of $L(M)$ are: $b, bb, baab, abab, abbbba, bbbabbbabbbabb$

Problem 2

1. Let $M = (K, \Sigma, \delta, s, F)$be a deterministic finite automaton. Under exactly what conditions $e \in L(M)$?

Solution

$$e \in L(M) \quad \text{iff} \quad s \in F.$$

2. Let $M = (K, \Sigma, \Delta, s, F)$ be a non-deterministic finite automaton. Under exactly what conditions $e \in L(M)$?

Solution Now we have two possibilities: $s \in F$ (computation of length 0) or there is a computation of length > 0 from (s, e) to (q, e) for $q \in F$ when $s \notin F$.

Problem 3 Let

$$M = (K,\ \Sigma,\ s,\ \Delta,\ F\ )$$

for $K = \{q_0, q_1, q_2, q_3\}$, $s = q_0$
$\Sigma = \{a, b\}$, $F = \{q_1, q_2, q_3\}$ and

$$\Delta = \{(q_0, a, q_1), (q_0, b, q_3), (q_1, a, q_2), (q_1, b, q_1), (q_3, a, q_3), (q_3, b, q_2)\}$$

1. List some elements of $L(M)$.

Solution $a, b, aa, bb, aba, abbba$

2. Write a regular expression for the language accepted by M. Simplify the solution.

Solution

$$L(M) = ab^* \cup ab^*a \cup ba^* \cup ba^*b = ab^*(e \cup a) \cup ba^*(e \cup b).$$

3. Define a deterministic M' such that $M \approx M'$, i.e. $L(M) = L(M')$.

Solution We complete M do a deterministic M' by adding a TRAP state q_4 and put

$$\Delta' = \delta = \Delta \cup \{(q_2, a, q_4), (q_2, b, q_4), (q_4, a, q_4), (q_4, b, q_4)\}$$

Justify why $M \approx M'$.

Solution q_4 is a trap state, it does not influence $L(M)$.

Problem 4 Let M be defined as follows

$$M = (K,\ \Sigma,\ s,\ \Delta,\ F\ )$$

for $K = \{q_0, q_1, q_2, q_3, \}$, $s = q_0$
$\Sigma = \{a, b, c\}$, $F = \{q_0, q_2, q_3\}$ and
$\Delta = \{(q_0, abc, q_0), (q_0, a, q_1), (q_0, e, q_3), (q_1, bc, q_1), (q_1, b, q_2), (q_2, a, q_2), (q_2, b, q_3), (q_3, a, q_3)\}$.

Find the regular expression describing the $L(M)$. Explain your steps. Does $e \in L(M)$?

Solution

$$L = \alpha_1 \cup \alpha_2 \cup \alpha_3 \cup \alpha_4$$

where

$\alpha_1 = (abc)^*$ - loop on q_0,

$\alpha_2 = (abc)^*a(bc)^*ba^*$ - path from q_0 to q_2,

$\alpha_3 = (abc)^*a(bc)^*ba^*ba^*$ - path from q_0 to q_3 via q_2,

$\alpha_3 = (abc)^*a^*$ - path from q_0 directly to q_3

This is not the only solution.

Observe that $e \in L$ as $q_0 \in F$ and also $(q_0, e, q_3) \in \Delta$ and $q_3 \in F$.

This is not the only solution.

Write down (you can draw the diagram) an automata M' such that $M' \equiv M$ and M' is defined by the **BOOK definition**.

Solution

Solution We apply the "stretching" technique to M and the new M' is is as follows.

$$M' = (K \cup \{p_1, p_2, p_3\}\ \Sigma,\ s = q_0,\ \Delta',\ F' = F\ )$$

for $K = \{q_0, q_1, q_2\},\ s = q_0$
$\Sigma = \{a, b\},\ F = \{q_0, q_2, q_3\}$ and
$\Delta' = \{(q_0, a, q_1), (q_0, e, q_3), (q_1, b, q_2), (q_2, a, q_2), (q_2, b, q_3), (q_3, a, q_3)\}$
$\cup\{(q_0, a, p_1), (p_1, b, p_2), (p_2, c, q_0), (q_1, b, p_3), (p_3, b, q_1)\}$.

Problem 5 For M defined as follows

$$M = (K,\ \Sigma,\ s,\ \Delta,\ F\ )$$

for $K = \{q_0, q_1, q_2, q_3\},\ s = q_0$
$\Sigma = \{a. b\},\ F = \{q_2\}$ and

$$\Delta = \{(q_0, a, q_3), (q_0, e, q_3), (q_0, b, q_1), q_0, e, q_1), (q_1, a, q_2), (q_2, b, q_3), (q_2, e, q_3)\}$$

Write 2 steps of the general method of transformation the NDFA M defined above into an equivalent DFA M'.

Step 1: Evaluate $\delta(E(q_0), a)$ and $\delta(E(q_0), b)$.

Step 2: Evaluate δ on all states that result from step 1.

Reminder: $E(q) = \{p \in K : (q, e) \overset{*,M}{\longmapsto} (p, e)\}$ and

$$\delta(Q, \sigma) = \bigcup\{E(p) : \exists q \in Q,\ (q, \sigma, p) \in \Delta\}.$$

Solution Step 1: First we need to evaluate $E(q)$, for all $q \in K$.

$$E(q_0) = \{q_0, q_1, q_3\} = S,\ E(q_1) = \{q_1\},\ E(q_2) = \{q_2 q_3\} \in F,\ E(q_3) = \{q_3\}$$

$$\delta(E(q_0), a) = \delta(\{q_0, q_1, q_3\}, a) = E(q_3) \cup E(q_2) \cup \emptyset = \{q_2, q_3\} \in F$$

$$\delta(E(q_0), b) = \delta(\{q_0, q_1, q_3\}, b) = E(q_1) \cup \emptyset \cup \emptyset = \{q_1\}$$

Solution Step 2:

$$\delta(\{q_2, q_3\}, a) = \emptyset \cup \emptyset = \emptyset$$

$$\delta(\{q_2, q_3\}, b) = E(q_3) \cup \emptyset = \{q_3\}$$

$$\delta(\{q_1\}, a) = E(q_2) = \{q_2, q_3\} \in F$$

$$\delta(\{q_1\}, b) = \emptyset$$