1. All infinite sets have different cardinality.
   **Justify:** The sets \( N \) (natural numbers) and \( Z \) (integers) have the same cardinality: \(|N| = |Z| = \aleph_0\). This is not the only example.

2. Regular language is a regular expression.
   **Justify:** By definition: "A language \( L \) is regular iff there is a regular expression \( \alpha \) such that \( L = L(\alpha) \).

3. \( L^+ = \{w_1...w_n : w_i \in L, i = 1, 2, ..., n, n \geq 2\} \).
   **Justify:** the correct condition is \( n \geq 1 \).

4. \( L^+ = L^* - \{e\} \).
   **Justify:** It holds only when \( e \not\in L \). When \( e \in L \) we get that \( e \in L^+ \) and \( e \not\in L^* - \{e\} \).

5. Let \( \alpha = (\emptyset^* \cap b^*) \cup \emptyset^* \). The language defined by \( \alpha \) is empty.
   **Justify:** \( L = L(\alpha) = (\{e\} \cap \{b\}^*) \cup \{e\} \cup \{e\} = \{e\} \neq \emptyset \)

6. A configuration of any finite automaton \( M = (K, \Sigma, \Delta, s, F) \) is any element of \( K \times \Sigma^* \times K \).
   **Justify:** it is element of \( K \times \Sigma^* \)

7. Let \( M \) be a finite state automaton, \( L(M) = \{\omega \in \Sigma^* : (s, \omega) \xrightarrow{^*M} (q, e)\} \).
   **Justify:** only when \( q \in F \)

8. For any \( M, L(M) \neq \emptyset \) if and only if the set \( F \) of its final states is non-empty.
   **Justify:** Let \( M \) be such that \( \Sigma = \emptyset, F \neq \emptyset, s \not\in F \), we get \( L(M) = \emptyset \).

9. The set \( F \) of final states of any non-deterministic finite automaton is always non-empty
   **Justify:** The definition says that \( F \) is a finite set, i.e. can be empty, hence for some \( M, L(M) = \emptyset \)

10. DFA and NDFA recognize the same class of languages.
    **Justify:** theorem proved in class

2. Two definitions of a non-deterministic automaton

   **BOOK DEFINITION:** \( M = (K, \Sigma, \Delta, s, F) \) is non-deterministic when
   \[ \Delta \subseteq K \times (\Sigma \cup \{e\}) \times K \]

   **OBSERVE** that \( \Delta \) is always finite because \( K, \Sigma \) are finite sets.

   **LECTURE DEFINITION:** \( M = (K, \Sigma, \Delta, s, F) \) is non-deterministic when \( \Delta \) is finite and
   \[ \Delta \subseteq K \times \Sigma^* \times K \]

   **OBSERVE** that we have to say in this case that \( \Delta \) is finite because \( \Sigma^* \) is infinite.

   **SOLVING PROBLEMS** you can use any of these definitions.
3 Very short questions (15pts)

For the QUESTIONS below do the following.

1. Draw the DIAGRAM

2. Determine whether it defines a finite state automaton.

3. Determine whether it is a deterministic / non-deterministic automaton.

4. Describe the language by writing a regular expression \( \delta(q_1, a) = q_2 \), that defines it.

Q1 Solution: \( M = (K, \Sigma, s, \Delta, F) \) for \( K = \{ q_0 \} = F \), \( s = q_0 \), \( \Sigma = \emptyset \), \( \Delta = \emptyset \).

\( M \) is deterministic and \( L(M) = \{ \epsilon \} \neq \emptyset \)

Q2 Solution: \( M = (K, \Sigma, s, \Delta, F) \) for \( \Sigma = \{ a, b \} \), \( K = \{ q_0, q_1 \} \), \( s = q_0 \), \( F = \{ q_0 \} \), \( \Delta = \{ (q_0, a, q_1), (q_0, b, q_0) \} \).

\( M \) is non deterministic; \( \Delta \) is not a function on \( K \times \Sigma \).

\( L(M) = (ab)^* \)

Q3 Solution: \( M = (K, \Sigma, s, \Delta, F) \) for \( \Sigma = \{ a, b \} \), \( K = \{ q_0, q_1, q_2, q_3 \} \), \( s = q_0 \), \( F = \{ q_0 \} \) and \( q_3 \) is a trap state.

\( \Delta = \{ (q_0, a, q_1), (q_0, b, q_1), (q_1, a, q_1), (q_1, e, q_2), (q_2, ab, q_2) \} \).

It is NOT an automaton. It has no initial state.

4 Problems

PROBLEM 1 (20pts)

Given an automata

\( M = (K, \Sigma, \delta, s, F) \)

such that \( \Sigma = \{ a, b \} \), \( K = \{ q_0, q_1, q_2, q_3 \} \), \( s = q_0 \), \( F = \{ q_0 \} \) and \( q_3 \) is a trap state.

We define \( \delta \) on non-trap states as follows.

\( \delta(q_0, a) = q_1 \), \( \delta(q_0, b) = q_0 \), \( \delta(q_1, a) = q_2 \), \( \delta(q_1, b) = q_1 \), \( \delta(q_2, a) = q_0 \)

Solution

We define \( \delta \) on the trap state \( q_3 \) as follows.

\( \delta(q_2, b) = q_3 \), \( \delta(q_3, a) = q_3 \), \( \delta(q_3, b) = q_3 \)

\( e, bbb, abbaa, abbaabbaa \in L(M) \), and \( a, aa, aba, baba \notin L(M) \)

Language of \( M \) is:

\( L(M) = b^* \cup (b^*ab^*aa)^* \)

Observe that \( e \in L(M) \) as the initial state is also a final state and \( b^* \in L(M) \)

\( b^*ab^*aa \in L(M) \) and then all repetitions of of this are in \( L(M) \) as a loop on \( q_0 \)
PROBLEM 2 (20pts)

Let

\[ M = (K, \Sigma, s, \Delta, F) \]

for \( K = \{q_0\}, s = q_0, \Sigma = \{a, b\}, F = \{q_0\} \) and

\[ \Delta = \{(q_0, aba, q_0), (q_0, ab, q_0)\} \]

Solution

1. List some elements of \( L(M) \).

\[ e, ab, abab, ababa, ababaaba, ... \]

2. Write a regular expression for the language accepted by \( M \).

\[ L = (ab \cup aba)^* \]

PROBLEM 3 (25pts)

Let \( M \) be defined as follows

\[ M = (K, \Sigma, s, \Delta, F) \]

for \( K = \{q_0, q_1, q_2, q_3\}, s = q_0 \)
\( \Sigma = \{a, b, c\}, F = \{q_0, q_2, q_3\} \) and

\[ \Delta = \{(q_0, abc, q_0), (q_0, a, q_1), (q_0, c, q_3), (q_1, bc, q_1), (q_1, b, q_2), (q_2, a, q_2), (q_2, b, q_3), (q_3, a, q_3)\} \]

1. Draw the state diagram of \( M \).

2. Find the regular expression describing the \( L(M) \). Explain your steps.

\[ L = \alpha_1 \cup \alpha_2 \cup \alpha_3 \cup \alpha_4 \]

where

\[ \alpha_1 = (abc)^* \] - loop on \( q_0 \),

\[ \alpha_2 = (abc)^*a(bc)^*ba^* \] - path from \( q_0 \) to \( q_2 \),

\[ \alpha_3 = (abc)^*a(bc)^*ba^* \] - path from \( q_0 \) to \( q_3 \) via \( q_2 \),

\[ \alpha_3 = (abc)^*a^* \] - path from \( q_0 \) directly to \( q_3 \)

This is not the only solution.

Observe that \( e \in L \) as \( q_0 \in F \) and also \((q_0, e, q_3) \in \Delta \) and \( q_3 \in F \).

This is not the only solution.

3. DRAW then DIAGRAM of an automata \( M' \) such that \( M' \equiv M \) and \( M' \) is defined by the BOOK definition.
We apply the "stretching" technique to $M$ and the new $M'$ COMPONENTS are as follows.

$$M' = (K \cup \{p_1, p_2, p_3\}, \Sigma, s = q_0, \Delta', F' = F)$$

for $K = \{q_0, q_1, q_2\}$, $s = q_0$

$\Sigma = \{a, b\}$, $F = \{q_2, q_3\}$ and

$\Delta' = \{(q_0, a, q_1), (q_0, e, q_3), (q_1, b, q_2), (q_2, a, q_2), (q_2, b, q_3), (q_3, a, q_3)\}$

$\cup \{(q_0, a, p_1), (p_1, b, p_2), (p_2, c, q_0), (q_1, b, p_3), (p_3, b, q_1)\}$.

**EXTRA CREDIT (10pts)**

For $M$ defined as follows

$$M = (K, \Sigma, s, \Delta, F)$$

for $K = \{q_0, q_1, q_2, q_3\}$, $s = q_0$

$\Sigma = \{a, b\}$, $F = \{q_2, q_3\}$ and

$\Delta = \{(q_0, a, q_1), (q_0, e, q_3), (q_0, b, q_2), (q_1, b, q_3), (q_1, e, q_3), (q_2, b, q_2), (q_2, e, q_3), (q_3, a, q_3)\}$

1. Write 4 steps of the general method of transformation the NDFA $M$, into an equivalent deterministic $M'$.

2. Draw the State Diagram of $M'$ thus far constructed.

   Reminder: $E(q) = \{p \in K : (q, e) \xrightarrow{*M} (p, e)\}$ and
   $$\delta(Q, \sigma) = \bigcup \{E(p) : \exists q \in Q, (q, \sigma, p) \in \Delta\}.$$

**Step 1:**

$$E(q_0) = \{q_0, q_1, q_3\}, E(q_1) = \{q_1, q_3\}, E(q_2) = \{q_2, q_3\}, E(q_3) = \{q_3\}.$$  

**Step 2:**

$$\delta(E(q_0), a) = \delta(\{q_0, q_1, q_3\}, a) = E(q_1) \cup E(q_3) = \{q_1, q_3\} \in F,$$

$$\delta(E(q_0), b) = \delta(\{q_0, q_1, q_3\}, b) = E(q_2) \cup E(q_3) \cup \emptyset = \{q_2, q_3\} \in F.$$  

**Step 3:**

$$\delta(\{q_1, q_3\}, a) = \emptyset \cup E(q_3) = \{q_3\} \in F,$$

$$\delta(\{q_1, q_3\}, b) = E(q_3) \cup \emptyset = \{q_3\} \in F,$$

$$\delta(\{q_2, q_3\}, a) = \emptyset \cup E(q_3) = \{q_3\} \in F,$$

$$\delta(\{q_2, q_3\}, b) = E(q_2) \cup \emptyset = \{q_2, q_3\} \in F.$$  

**Step 4:**

$$\delta(\{q_3\}, a) = E(q_3) = \{q_3\} \in F, \quad \delta(\{q_3\}, b) = \emptyset,$$

$$\delta(\emptyset, a) = \emptyset, \quad \delta(\emptyset, b) = \emptyset$$

End of the construction.

You must Draw the DIAGRAM.