1 YES/NO questions

1. For any function $f$ from $A \neq \emptyset$ onto $A$, $f$ has property $f(a) \neq a$ for certain $a \in A$.
   **Justify:** $f(x) = x$ is always "onto".

2. For any binary relation $R \subseteq A \times A$, $R^{-1}$ exists.
   **Justify:** The set $R^{-1} = \{(b, a) : (a, b) \in R\}$ always exists.

3. All infinite sets have the same cardinality.
   **Justify:** $|N| < |2^N|$ by Cantor Theorem and $N, 2^N$ are infinite.

4. Set $A$ is uncountable iff $R \subseteq A$ ($R$ is the set of real numbers).
   **Justify:** $R, 2^R$ are both uncountable and $R$ is not a subset of $2^R$ ($R \not\subseteq 2^R$) but $R \in 2^R$.

5. Let $A \neq \emptyset$ such that there are exactly 25 partitions of $A$. It is possible to define 20 equivalence relations on $A$.
   **Justify:** one can define up to 25 (as many as partitions) of equivalence classes.

6. There is a relation that is equivalence and order at the same time.
   **Justify:** equality relation.

7. Let $A = \{n \in N : n^2 + 1 \leq 15\}$. It is possible to define 8 alphabets $\Sigma \subseteq A$.
   **Justify:** $A$ has 4 elements, so we have $2^4 > 8$ subsets.

8. There is exactly as many languages over alphabet $\Sigma = \{a\}$ as real numbers.
   **Justify:** $|\Sigma^*| = \aleph_0, |2^{\Sigma^*}| = |R| = C$.

9. Let $\Sigma = \{a, b\}$, there are more than 20 words of length 4 over $\Sigma$.
   **Justify:** There are exactly $2^4 = 16$ words of length 4 over $\Sigma$ and $16 < 20$.

10. $L^* = \{w_1...w_n : w_i \in L, i = 1, 2,.., n \geq 1\}$.
    **Justify:** $n \geq 0$.
    $L^+ = LL^*$.
    **Justify:** the problem is only with cases $e \in L$ or $e \not\in L$. When $e \in L$, then $e \in L^+$, and always $e \in L^*$, hence $e \in LL^*$.
        When $e \not\in L$, then $e \not\in L^+$, and always $e \in L^*$, hence $e \in LL^*$ and $L^+ \neq LL^*$

11. $L^+ = L^* - \{e\}$.
    **Justify:** only when $e \not\in L$. When $e \in L$ we get that $e \in L^+$ and $e \not\in L^* - \{e\}$.

12. If $L = \{w \in \{0, 1\}^* : w$ has an unequal number of 0’s and 1’s $\}$, then $L^* = \{0, 1\}^*$.
    **Justify:** $1 \in L, 0 \in L$ so $\{0, 1\} \subseteq L \subseteq \Sigma^*$, hence $\{0, 1\}^* \subseteq L^* \subseteq (\Sigma^*)^* = \Sigma^* = \{0, 1\}^*$ and $L^* = \{0, 1\}^*$.
13. For any languages \( L_1, L_2 \), \( L_1^* = L_2^* \) if and only if \( L_1 = L_2 \).

**Justify:** Consider \( L_1 = \{a, e\}, L_2 = \{a\} \). Obviously, \( L_1 \neq L_2 \) and \( L_1^* = L_2^* \).

14. For any languages \( L_1, L_2 \), \((L_1 \cup L_2) \cap L_1 = L_1\).

**Justify:** languages are sets and \((A \cup B) \cap A = A\).

15. For any languages \( L_1, L_2 \), \((L_1 \cup L_2)^* = L_1^*\).

**Justify:** languages are sets so it is true only when \( L_1 \subseteq L_2 \).

16. \((\emptyset \cap a) \cup b^*\) describes a language with only one element.

**Justify:** \( \emptyset \cup b^* = b^*, b^* \cap \{e\} = \{e\} \)

17. \((\emptyset \cap a) \cup b^*\) \(a^*\) is a finite regular language.

**Justify:** \( b^* \cap a^* = \{e\} = \emptyset^* \)

18. \((\emptyset \cup \{e\}) \cap \{ab\}^*\) is a finite regular language.

**Justify:** \((\emptyset \cup \{e\}) \cap \{ab\}^* = \{a, e\} \cap \{ab\}^* = \{e\} = \emptyset^* \)

19. Any regular language has a finite description.

**Justify:** by definition \( L = \mathcal{L}(r) \) and \( r \) is a finite string.

20. Any finite language is regular.

**Justify:** \( L = \{w_1\} \cup \ldots \cup \{w_1\} \) and \( \{w_1\} \) has a finite description \( w_i \)

21. Every deterministic automata is also non-deterministic.

**Justify:** \( K \times \Sigma \subseteq K \times \Sigma \cup \{e\} \subseteq \Sigma^* \) and any function is a relation

The set of all configurations of any non-deterministic state automata is always non-empty.

**Justify:** \( K \neq \emptyset \), because \( s \in K \). If \( \Sigma = \emptyset \) the set of all configuration of non-deterministic automata (book definition) is a subset of \( K \times \emptyset \cup \{e\} \neq \emptyset \) as it always contains \((s, e)\). For the lecture definition, the set of all configuration is a subset of \( K \times \Sigma^* \) and always \( e \in \Sigma^* \) hence always \((s, e) \in K \times \Sigma^* \).

22. Let \( M \) be a finite state automaton, \( L(M) = \{w \in \Sigma^*: (q, w) \xrightarrow{\Sigma^*} (s, e)\} \).

**Justify:** \( L(M) = \{w \in \Sigma^*: \exists q \in F((s, w) \xrightarrow{\Sigma^*} (q, c))\} \)

23. For any automata \( M, L(M) \neq \emptyset \).

**Justify:** if \( F = \emptyset, L(M) = \emptyset \)

24. \( L(M_1) = L(M_2) \) iff \( M_1, M_2 \) are deterministic.

**Justify:** Let \( M_1 \) be an automata over \( \{a, b\} \) with with \( \Delta = \{(q_0, ab, q_0)\}, F = \{q_0\}, s = q_0 \) and let \( M_2 \) be an automata over \( \{a, b\} \) with with \( \Delta = \{(q_0, ab, q_0), (q_0, e, q_1)\}, F = \{q_1\}, s = q_0 \). \( L(M_1) = L(M_2) = (ab)^* \) and both are non-deterministic

25. DFA and NDFA recognize the same class of languages.

**Justify:** basic theorem

26. Let \( M_1 \) be a deterministic, \( M_2 \) be a nondeterministic FA, \( L_1 = L(M_1) \) and \( L_2 = L(M_2) \) then there is a deterministic automaton \( M \) such that \( L(M) = (L^* \cup (L_1 - L_2^*))L_1 \)

**Justify:** the class of finite automata is closed under \( *, \cup, -, \cap \)
TWO DEFINITIONS OF NON DETERMINISTIC AUTOMATA

BOOK DEFINITION: \( M = (K, \Sigma, \Delta, s, F) \) is non-deterministic when
\[
\Delta \subseteq K \times (\Sigma \cup \{e\}) \times K
\]

OBSERVE that \( \Delta \) is always finite because \( K, \Sigma \) are finite sets.

LECTURE DEFINITION: \( M = (K, \Sigma, \Delta, s, F) \) is non-deterministic when \( \Delta \) is finite and
\[
\Delta \subseteq K \times \Sigma^* \times K
\]

OBSERVE that we have to say in this case that \( \Delta \) is finite because \( \Sigma^* \) is infinite.

SOLVING PROBLEMS you can use any of these definitions.

2 Very short questions

For all Questions below do the following.

1. Draw the State Diagram of \( M \).
2. Determine whether \( M \) is/it is not a finite state automata and determine whether \( M \) is deterministic or non-deterministic, if applicable.
3. Describe \( L(M) \) by writing a regular expression that defines it.

Q1 \( M = (K, \Sigma, s, \Delta, F) \) for \( K = \{q_0\} = F, s = q_0, \Sigma = \emptyset, \Delta = \emptyset \).
Solution: \( M \) is deterministic and
\[
L(M) = \{e\} \neq \emptyset
\]

Q2 \( M = (K, \Sigma, s, \Delta, F) \) for \( \Sigma = \{a,b\}, K = \{q_0, q_1\}, s = q_0, F = \{q_0\}, \Delta = \{(q_0, a, q_1), (q_1, b, q_0)\} \).
Solution: \( M \) is non deterministic; \( \Delta \) is not a function on \( K \times \Sigma \).
\[
L(M) = (ab)^*
\]

Q3 \( M = (K, \Sigma, s, \Delta, F) \) for \( \Sigma = \{a,b\}, K = \{q_0, q_1, q_2\}, F = \{q_1\}, \Delta = \{(q_0, a, q_1), (q_0, b, q_1), (q_1, a, q_1), (q_1, e, q_2), (q_2, ab, q_2)\} \).
Solution: It is NOT an automaton. It has no initial state.

Q4 \( M = (K, \Sigma, s, \Delta, F) \) for \( \Sigma = \{a,b\}, K = \{q_0, q_1, q_2, q_3\}, s = q_0, F = \emptyset, \Delta = \{(q_0, a, q_1), (q_1, b, q_2), (q_2, a, q_1), (q_0, e, q_3), (q_2, a, q_3)\} \).
Solution: \( M \) is non deterministic; \( \Delta \subseteq K \times \Sigma \cup \{e\} \times K \).
\[
L(M) = \emptyset
\]
3 Problems

In Problems 1-5 solutions you have to DRAW DIAGRAMS; do not need to LIST the Components

Problem 1 Let $L$ be a language defines as follows

$$L = \{ w \in \{a,b\}^* : \text{every } a \text{ is either immediately proceeded or followed by } b \}.$$ 

1. Describe a regular expression $r$ such that $L(r) = L$ (Meaning of $r$ is $L$).

Solution $L = (b \cup ab \cup ba \cup bab)^*$

2. Construct a finite state automata $M$, such that $L(M) = L$.

Solution

Components of $M$ are:

$$K = \{s\}, \{a,b\}, \ s, \ F = \{s\},$$

$$\Delta = \{(s,b,s), (s,ab,s), (s,ba,s), (s,bab,s)\}.$$ 

Some elements of $L(M)$ are: $b, bb, baab, abab, abbbba, bbabbbabbbabb$

Problem 2

1. Let $M = (K, \Sigma, \delta, s, F)$ be a deterministic finite automaton. Under exactly what conditions $e \in L(M)$?

Solution

$$e \in L(M) \iff s \in F.$$ 

2. Let $M = (K, \Sigma, \Delta, s, F)$ be a non-deterministic finite automaton. Under exactly what conditions $e \in L(M)$?

Solution Now we have two possibilities: $s \in F$ (computation of length 0) or there is a computation of length $> 0$ from $(s,e)$ to $(q,e)$ for $q \in F$ when $s \not\in F$.

Problem 3 Let

$$M = (K, \Sigma, s, \Delta, F)$$

for $K = \{q_0, q_1, q_2, q_3\}, \ s = q_0$

$\Sigma = \{a, b\}, \ F = \{q_1, q_2, q_3\}$ and

$$\Delta = \{(q_0, a, q_1), (q_0, b, q_3), (q_1, a, q_2), (q_1, b, q_1), (q_3, a, q_3), (q_3, b, q_2)\}.$$ 

1. List some elements of $L(M)$.

Solution $a, b, aa, bb, aba, abba$

2. Write a regular expression for the language accepted by $M$. Simplify the solution.

Solution

$$L(M) = ab^* \cup ab^*a \cup ba^* \cup ba^*b = ab^*(e \cup a) \cup ba^*(e \cup b).$$ 

3. Define a deterministic $M'$ such that $M \approx M'$, i.e. $L(M) = L(M')$. 


Solution  We complete $M$ do a deterministic $M'$ by adding a TRAP state $q_4$ and put

$$\Delta' = \delta = \Delta \cup \{(q_2, a, q_4), (q_2, b, q_4), (q_4, a, q_4), (q_4, b, q_4)\}$$

Justify why $M \equiv M'$.

Solution  $q_4$ is a trap state, it does not influence $L(M)$.

Problem 4  Let $M$ be defined as follows

$$M = (K, \Sigma, s, \Delta, F)$$

for $K = \{q_0, q_1, q_2, q_3\}, s = q_0$

$\Sigma = \{a, b, c\}, F = \{q_0, q_2, q_3\}$ and

$$\Delta = \{(q_0, abc, q_0), (q_0, a, q_1), (q_0, c, q_3), (q_1, bc, q_1), (q_1, b, q_2), (q_2, a, q_2), (q_2, b, q_3), (q_3, a, q_3)\}.$$ 

Find the regular expression describing the $L(M)$. Explain your steps. Does $e \in L(M)$?

Solution

$$L = \alpha_1 \cup \alpha_2 \cup \alpha_3 \cup \alpha_4$$

where

$\alpha_1 = (abc)^* -$ loop on $q_0$,

$\alpha_2 = (abc)^*a(bc)^*ba^* -$ path from $q_0$ to $q_2$,

$\alpha_3 = (abc)^*a(bc)^*ba^* -$ path from $q_0$ to $q_3$ via $q_2$,

$\alpha_3 = (abc)^*a^* -$ path from $q_0$ directly to $q_3$

This is not the only solution.

Observe that $e \in L$ as $q_0 \in F$ and also $(q_0, e, q_3) \in \Delta$ and $q_3 \in F$.

This is not the only solution.

Write down (you can draw the diagram) an automata $M'$ such that $M' \equiv M$ and $M'$ is defined by the BOOK definition.

Solution

Solution  We apply the "stretching" technique to $M$ and the new $M'$ is as follows.

$$M' = (K \cup \{p_1, p_2, p_3\}, \Sigma, s = q_0, \Delta', F' = F)$$

for $K = \{q_0, q_1, q_2\}, s = q_0$

$\Sigma = \{a, b\}, F = \{q_0, q_2, q_3\}$ and

$$\Delta' = \{(q_0, a, q_1), (q_0, e, q_3), (q_1, b, q_2), (q_2, a, q_2), (q_2, b, q_3), (q_3, a, q_3)\}
\cup\{(q_0, a, p_1), (p_1, b, p_2), (p_2, c, q_0), (q_1, b, p_3), (p_3, b, q_1)\}.$$ 

Problem 5

For $M$ defined as follows

$$M = (K, \Sigma, s, \Delta, F)$$

for $K = \{q_0, q_1, q_2, q_3\}, s = q_0$

$\Sigma = \{a, b\}, F = \{q_2\}$ and

$$\Delta = \{(q_0, a, q_3), (q_0, e, q_3), (q_0, b, q_1), (q_0, e, q_1), (q_1, a, q_2), (q_2, b, q_3), (q_2, e, q_3)\}$$
Write 2 steps of the general method of transformation the NDFA $M$ defined above into an equivalent DFA $M'$.

**Step 1:** Evaluate $\delta(E(q_0), a)$ and $\delta(E(q_0), b)$.

**Step 2:** Evaluate $\delta$ on all states that result from step 1.

Reminder: $E(q) = \{p \in K : (q, e) \xrightarrow{M} (p, e)\}$ and

$$\delta(Q, \sigma) = \bigcup\{E(p) : \exists q \in Q, \ (q, \sigma, p) \in \Delta\}.$$  

**Solution Step 1:** First we need to evaluate $E(q)$, for all $q \in K$.

$$E(q_0) = \{q_0, q_1, q_3\} = S, \ E(q_1) = \{q_1\}, \ E(q_2) = \{q_2, q_3\} \in F, \ E(q_3) = \{q_3\}$$

$$\delta(E(q_0), a) = \delta(\{q_0, q_1, q_3\}, a) = E(q_3) \cup E(q_2) \cup \emptyset = \{q_2, q_3\} \in F$$

$$\delta(E(q_0), b) = \delta(\{q_0, q_1, q_3\}, b) = E(q_1) \cup \emptyset \cup \emptyset = \{q_1\}$$

**Solution Step 2:**

$$\delta(\{q_2, q_3\}, a) = \emptyset \cup \emptyset = \emptyset$$

$$\delta(\{q_2, q_3\}, b) = E(q_3) \cup \emptyset = \{q_3\}$$

$$\delta(\{q_1\}, a) = E(q_2) = \{q_2, q_3\} \in F$$

$$\delta(\{q_1\}, b) = \emptyset$$