PART 1 (10pts) YES/NO QUESTIONS - (2pts) each

1. \( L \subseteq \Sigma^* \) is context-free if and only if \( L = L(G) \)
   \textbf{Justify:} It is a definition of context-free language PROVIDED \( G \) is a context -free grammar
   \( n \)

2. \( L(G) = \{ w \in V : S \Rightarrow^* Gw \} \)
   \textbf{Justify:} \( w \in \Sigma^* \)
   \( n \)

3. Any regular language is accepted by a pushdown automaton
   \textbf{Justify:} Any regular language is accepted by a finite automata and any finite automata is a pushdown automata operating on an empty stock.
   \( y \)

4. Context-free languages are closed under intersection
   \textbf{Justify:} Take \( L_1 = a^b b^n c^m, L_2 = d^n b^p c^n \), both CF and we get that \( L_1 \cap L_2 = a^b b^n c^n \) is not CF
   \( n \)

5. The union of a context-free language and regular language is a context-free language
   \textbf{Justify:} regular language is also a context free language and context free languages are closed under union
   \( y \)

6. Every subset of a regular language is a language.
   \textbf{Justify:} a subset of a set is a set.
   \( y \)

7. Any context -free language is accepted by some PD automata
   \textbf{Justify:} Main Theorem for Context Free Languages
   \( y \quad n \)

QUESTION 1 (8pts)

1. Construct a context-free grammar \( G \) such that
   \[ L(G) = \{ w \in \{a, b\}^* : w = w^R \}. \]

   \textbf{Solution} \( G = (V, \Sigma, R, S) \), where
   \[ V = \{a, b, S\}, \quad \Sigma = \{a, b, \}, \]
   \[ R = \{ S \rightarrow aSa \mid bSb \mid a \mid b \mid e \}. \]

2. Give an example of a derivation of a word \( ababa \) in \( G \)

   \textbf{Derivation example}
   \[ S \Rightarrow aSa \Rightarrow abSba \Rightarrow ababa \]

3. Write a proof that the word \( ababa \in L(G) \) fulfills the required property \( w = w^R \)

   \textbf{Proof}
   We know that \( (xy)^R = y^R x^R \) and \( (xyz)^R = z^R (xy)^R = z^R y^R x^R \)
   We evaluate
   \[ (ababa)^R = ((ab) a (ba))^R = (ba)^R a^R (ab)^R = ababa \]
QUESTION 2 (7pts)

1. Construct a **pushdown** automaton $M$ such that

$$L(M) = \{w \in \{a, b\}^* : w = w^R\}$$

JUST DRAW A DIAGRAM. Here ARE the Components

**Solution** $M = (K, \Sigma, \Gamma, \Delta, s, F)$, where

$$K = \{s, f\}, \quad \Sigma = \{a, b\}, \quad \Gamma = \{a, b\}, \quad F = \{f\}, \quad s = s,$$

$$\Delta = \{((s, a, e), (s, a)), ((s, b, e), (s, b)), ((s, e, e), (f, e)), ((s, a, e), (f, e)), ((s, b, e), (f, e)), ((f, a, a), (f, e)), ((f, b, b), (f, e))\}$$

![Diagram](image)

2. Trace a transitions of $M$ that lead to the acceptance of the string *ababa*.

**Solution**

```
s  ababa  e
s  baba  a
s  aba  ba
f  ba  ba
f  a  a
f  e  e
```
Proof - GENERAL CASE for Question 1

1. Construct a context-free grammar $G$ such that

$$L(G) = \{ w \in \{a, b\}^* : w = w^R \}.$$ 

Solution

$G = (V, \Sigma, R, S)$, where

$$V = \{a, b, S\}, \quad \Sigma = \{a, b\},$$

$$R = \{ S \to aSa | bSb | a | b | \varepsilon \}.$$

Here is a general case- for you to study (we did it in class!)

Grammar correctness justification

Observation  We proved in class that for any $x, y \in \Sigma^*$, $(xy)^R = y^R x^R$. From this we have that

$$(xyz)^R = ((xy)z)^R = z^R (xy)^R = z^R y^R x^R$$

observe that the rules $S \to aSa | bSb | \varepsilon$ generate the language $L_1 = \{ww^R : w \in \Sigma^*\}$. With additional rules $S \to a | b$ we get hence the language $L = L_1 \cup \{ww^R : w \in \Sigma^*\} \cup \{wbw^R : w \in \Sigma^*\}$. Now we are ready to prove that

$$L = L(G) = \{ w \in \{a, b\}^* : w = w^R \}.$$  

Proof  Let $w \in L$, i.e. $w = xx^R$ or $w = xax^R$ or $w = xb x^R$. We show that in each case $w = w^R$ as follows.

c1:  $w^R = (xx^R)^R = (x^R)^R x^R = xx^R = w$ (used property: $(x^R)^R = x$).

c2:  $w^R = (xax^R)^R = (x^R)^R a^R x^R = xax^R = w$ (used Observation 1 and properties: $(x^R)^R = x$ and $a^R = a$).

c3:  $w^R = (xbx^R)^R = (x^R)^R b^R x^R = xb x^R = w$ (used Observation and properties: $(x^R)^R = x$ and $b^R = b$).