PART 1: YES/NO QUESTIONS (10pts)

Circle your answer and write a short justification (1pt)

No justification- no points

1. \(\{a, \{\emptyset\}, 2\} \cap 2^\emptyset = \emptyset\)
   
   Justify: \(2^\emptyset = \{\emptyset\}\) and \(\emptyset \not\in \{a, \{\emptyset\}, 2\}\)

2. The relation \(R = \{(n, m) : n \in N \text{ and } m = 1\}\) is a function
   
   Justify: \(R\) is a function defined by the formula is \(R(n) = 1\) for all \(n \in N\)

3. If the \(A\) is uncountable, then \(|A| = C\)
   
   Justify: The set \(2^R\) of all subsets of real numbers \(R\) is uncountable, but by Cantor Theorem
   we have that \(|2^R| > |R| = C|\).

4. The set \(A = \{x \in N : x \geq 0\}\) is finite
   
   Justify: \(A = \{x \in N : x \geq 0\} = N\) and \(N\) is countably infinite

5. The set \(\{\emptyset, \{\emptyset\}, 1, 2\}\) is an alphabet
   
   Justify: \(A\) is a finite set and any finite set is an alphabet by definition

6. Let \(\Sigma = \{\emptyset\}\). There are uncountably many languages over \(\Sigma\)
   
   Justify: \(\Sigma \neq \emptyset\) as \(\emptyset\) is element of \(\Sigma\). Hence there are infinitely countably many elements
   in \(\Sigma^*\) and uncountably many subsets of \(\Sigma^*\). In fact exactly as many as real numbers as
   \(|2^{\Sigma^*}| = |R| = C|\).

7. For any languages \(L_1, L_2, L\) over \(\Sigma \neq \emptyset\)
   \((L_1 \cap L_2) \cup L = (L_1 \cup L) \cap (L_2 \cup L)\)
   
   Justify: Languages are sets and this is the Law of Distributivity of union over intersection
   of sets

8. \(L^* = \{w_1...w_n : w_i \in L, i = 1, 2,...n, n \geq 1\}\)
   
   Justify: This is definition of \(L^+\); Kleene Star must have condition
   \(n \geq 0\).

9. Regular language is a regular expression.
   
   Justify: Regular language is defined by a regular expression.

10. For any language \(L\) over an alphabet \(\Sigma, L^+ = L \cup L^*\).
    
    Justify: holds only only when \(e \in L\) as \(e \in L^*\)
PART 2

QUESTION 1 (7pts)
Given an alphabet $\Sigma = \{a, b\}$ and a regular expression $\alpha = a^*b \cup (a \cup b)^*$. 
1. (3pts) Evaluate $L = L(\alpha)$.

Solution
1. We evaluate

$$L = L(a^*b \cup (a \cup b)^*) = L(a^*)L(b) \cup (L(a) \cup L(b))^* = \{a\}^*\{b\} \cup \{a, b\}^*$$

2. (4pts) Give a property describing the language $L$ determined by $\alpha$

Solution
Observe that $\{a\}^*\{b\} \subseteq \{a, b\}^*$ hence the language is

$$L = \{w : w \in \{a, b\}^*\} = \Sigma^*$$

QUESTION 2 (8pts)
Let $\Sigma = \{a, b\}$ and a language $L \subseteq \Sigma^*$ be defined as follows:

$$L = \{w \in \Sigma^* : w \text{ contains less then two a's}\}$$

Write a regular expression $\alpha$, such that $L(\alpha) = L$. Use shorthand notation. Explain shortly your answer.

Solution

$$\alpha = b^* \cup b^*ab^*$$

Explanation

"$w$ contains less then two a’s" means "contains zero or one a"

$b^*$ contains no of a (case n=0)

$b^*ab^*$ contains one occurrence of a (case n=1)