PART 1: YES/NO QUESTIONS

1. \( \{2, \{\emptyset\}, \emptyset\} \cap 2^\emptyset = \emptyset \)
   Justify: \( 2^\emptyset = \{\emptyset\} \) and \( \{2, \{\emptyset\}, \emptyset\} \cap \emptyset \neq \emptyset \) as \( \emptyset \in \{2, \{\emptyset\}, \emptyset\} \)

2. The relation \( R = \{(n, m) : n \in \mathbb{N} \text{ and } m \in \mathbb{N} \setminus \{1\}\} \) is a function
   Justify: For example: \( (0, 0) \in R \) and \( (0, 2) \in R \), \( (0, 3) \in R \) so \( R \) is not a function

3. A set \( A = \{(x, y) \in \mathbb{N} \times \mathbb{N} : x^2 + 1 < 1, y^2 \geq 0\} \) is countable
   Justify: \( A = \emptyset \times \mathbb{N} = \emptyset \) is finite, so is countable

4. The set \( A = \{\emptyset, \{\emptyset\}\} \) is an alphabet
   Justify: \( A \) has two elements so it is a finite set

5. Let \( \Sigma = \emptyset \). There are countably many languages over \( \Sigma \)
   Justify: \( \Sigma^* = 0^* = \{e\} \). We have that and \( \emptyset \subseteq \{e\} \) and \( \{e\} \subseteq \{e\} \). Hence there are two languages \( L_1 = \emptyset \) and \( L_1 = \{e\} \) over \( \Sigma \)

6. Let \( \Sigma = \{\emptyset\} \). There are countably many finite languages over \( \Sigma \)
   Justify: \( \Sigma^* \) in infinitely countable. Finite language is a finite subset of \( \Sigma^* \). We have a Theorem that says that the set of all finite subsets of any infinitely countable set is countably infinite. By definition a set is countable if is finite or infinitely countable, hence there countably many finite languages over \( \Sigma \)

7. For any languages \( L_1, L_2, L \) over \( \Sigma \neq \emptyset \)
   \( \Sigma^* - (L_1 \cap L_2) = (\Sigma^* - L_1) \cup (\Sigma^* - L_2) \)
   Justify: Languages are sets so de Morgan Laws hold for them

8. \( L^* = \{w_1...w_n : w_i \in L, i = 1, 2,..n, n \geq 0\} \)
   Justify: Definition of Kleene Star operation on languages

9. Regular language is a regular expression.
   Justify: A language \( L \subseteq \Sigma^* \) is regular if and only if \( L \) is represented by a regular expression, i.e. when there is \( \alpha \in \mathcal{R} \) such that \( L = \mathcal{L}(\alpha) \)

10. Let \( \alpha = (a \cup b)^*b \) be a regular expression over \( \Sigma = \{a, b\} \).
    \( L = \mathcal{L}(\alpha) = \{w \in \{a, b\}^* : w \text{ ends with } b\} \)
    Justify: \( \mathcal{L}((a \cup b)^*b) = (\{a\} \cup \{b\})^*\{b\} = \{a,b\}^*\{b\} = \Sigma^*\{b\} \)
PART 2

QUESTION 1 SOLUTION

Given an alphabet $\Sigma = \{a, b, c\}$ and a regular expression $\alpha = ((c^*a) \cup (bc^*))$

1. Write $\alpha$ in the shorthand notation:
   $$\alpha = c^*a \cup bc^*$$

2. Given the representation function $L : \mathcal{R} \rightarrow 2^{\Sigma^*}$. Write all steps in evaluation of $L(\alpha) = L$
   $L(\alpha) = L(c^*a \cup bc^*) = \mathcal{L}(c^*a) \cup \mathcal{L}(bc^*) = L(c^*)L(a) \cup L(b)L(c^*) = \{c\}*\{a\} \cup \{b\}\{c\}* = L$

3. List elements of the infinite set $L = \alpha$ (shorthand notation)
   $$L = \{c\}*\{a\} \cup \{b\}\{c\}* = \{e, c, cc, \ldots\}\{a\} \cup \{b\}\{e, c, cc, \ldots\} = \{a, ca, cca, \ldots\} \cup \{b, bc, bcc, \ldots\}$$

4. Write a property $P(w)$ describing $L = \alpha$, i.e. write the language $L$ as $L = \{w \in \Sigma^* : P(w)\}$
   $$L = \{w \in \Sigma^* : w = a \text{ or } w = b \text{ or } w \text{ is any nonempty sequence of } c \text{'s preceded by } a, \text{ or followed by } b \}$$

QUESTION 2 SOLUTION

Let $\Sigma = \{a, b\}$. Let $L \subseteq \Sigma^*$ be defined as
   $$L = \{w \in \{a, b\}^* : w \text{ contains at most three } b \text{'s}\}$$

1. Write a regular expression $\alpha$, such that $L(\alpha) = L$. Use shorthand notation.
   $$\alpha = a^* \cup a^*ba^* \cup a^*ba^*ba^* \cup a^*ba^*ba^*ba^*$$

2. Explain your answer.
   $a^*$ contains 0 of $b$'s,
   $a^*ba^*$ contains 1 occurrence of $b$,
   $b^*aa^*ba^*$ contains 2 occurrence of $b$,
   $a^*ba^*ba^*ba^*$ contains 3 occurrence of $b$,